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# Combinatorial Auctions with Interdependent Valuations: SOS to the Rescue

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**Abstract.** We study combinatorial auctions with interdependent valuations, where each agent $i$ has a private signal $s_i$ that captures her private information and the valuation function of every agent depends on the entire signal profile, $s = (s_1, \dots, s_n)$. The literature in economics shows that the interdependent model gives rise to strong impossibility results and identifies assumptions under which optimal solutions can be attained. The computer science literature provides approximation results for simple single-parameter settings (mostly single-item auctions or matroid feasibility constraints). Both bodies of literature focus largely on valuations satisfying a technical condition termed *single crossing* (or variants thereof). We consider the class of *submodular over signals* (SOS) valuations (without imposing any single crossing-type assumption) and provide the first welfare approximation guarantees for multidimensional combinatorial auctions achieved by universally ex post incentive-compatible, individually rational mechanisms. Our main results are (i) four approximation for any single-parameter downward-closed setting with single-dimensional signals and SOS valuations; (ii) four approximation for any combinatorial auction with multidimensional signals and *separable*-SOS valuations; and (iii) $(k+3)$ and $(2\log(k)+4)$ approximation for any combinatorial auction with single-dimensional signals, with $k$-sized signal space, for SOS and strong-SOS valuations, respectively. All of our results extend to a parameterized version of SOS, *d-approximate SOS*, while losing a factor that depends on $d$.

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## 1. Introduction

The classic setting considered in auction design is that of independent private values IPV. In 1982, Milgrom and Weber [30] introduced the model of interdependent values IDV, building upon the seminal prior work of Wilson [42] on common values. In the model of interdependent values, an agent's value depends on information distributed among a wide set of agents. The importance of this model was recently celebrated in the 2020 Nobel Prize in Economics awarded to Milgrom and Weber [30] for "the improvement to auction theory and inventions of new auction formats".

In this setting, every agent has a private signal (or set of signals) $s_i$ that captures the information the agent has regarding the goods for sale. The valuation of every agent is a (known) function $v_i(s_1, \dots, s_n)$ of all the agents' private signals. A typical example of interdependent values is the weighted sum valuation (see, e.g., Myerson [31], Roughgarden and Talgam-Cohen [37]). In this example, $v_i(\mathbf{s}) = s_i + \beta\sum_{j\neq i} s_j$ for some $\beta \leq 1$. This type of valuation function captures settings where an agent's value depends both on how much he likes the item ($s_i$) and on the resale value, which is naturally estimated in terms of how much other agents like the item ($\sum_{j\neq i} s_j$).

Interdependent values capture important real-world settings, such as valuations for oil drilling rights, broadcast rights, artwork, and many more. Essentially, every setting that exhibits asymmetry of information amongst agents is a setting of interdependent values. This setting is much more realistic than the more standard model of independent private values. However, once we move from the standard model of independent values to interdependent values, the economics literature provides mainly strong impossibility results (Dasgupta and Maskin [15], Jehiel and Moldovanu [23]).

### 1.1. Maximizing Social Welfare

Consider the goal of maximizing social welfare in interdependent settings. Here, a direct revelation mechanism consists of each agent $i$ reporting a bid for their private signal $s_i$ and the auctioneer determining the allocation and payments. (It is assumed that the auctioneer knows the form of the valuation functions $v_i(\cdot)$.)

In interdependent settings, it is not possible (except perhaps in degenerate situations) to design dominant-strategy incentive-compatible (DSIC) auctions because an agent's value depends on *all* of the signals; if, say, agent $i$ misreports his signal, then agent $j$ might win at a price above her value if she reports truthfully. The next strongest equilibrium notion one could hope for is to maximize efficiency in ex post equilibrium; bidding truthfully is an *ex post equilibrium* if an agent does not regret having bid truthfully, given that other agents bid truthfully. In other words, bidding truthfully is a Nash equilibrium for every signal profile. Note that every ex post equilibrium is a Bayes–Nash incentive-compatible equilibrium but not necessarily vice versa, and therefore, ex post equilibria are much more robust; they do not depend on knowledge of the priors, and bidders need not think about how other bidders might be bidding. This increases our confidence that an ex post equilibrium is likely to be reached.

Maximizing social welfare with *private valuations* is a solved problem. The classical Vickrey–Clarke–Grove (VCG) family of mechanisms (Clarke [12], Groves [21], Vickrey [40]), of which the Vickrey second-price auction is a special case, is dominant strategy incentive compatible and guarantees optimal social welfare in general social choice settings. In the interdependent values setting, however, the situation is much more complex. Prior research on interdependent values considered the highly restricted single-parameter setting and even within this setting, only valuations satisfying a technical condition called "single crossing" (SC) (Athey [3], Ausubel [4], Bergemann et al. [7], Chawla et al. [10], Che et al. [11], Dasgupta and Maskin [15], d'Aspremont and Gérard-Varet [16], Li [27], Maskin [28], Milgrom and Weber [30], Roughgarden and Talgam-Cohen [37]). Unfortunately, the single-crossing condition does not generally suffice to obtain optimal social welfare in settings beyond that of a single-item auction with single-dimensional signals. It is insufficient in fairly simple settings, such as two-item, two-bidder auctions with unit-demand valuations (see Appendix A) or single-parameter settings with downward-closed feasibility constraints (see Appendix B).

Moreover, there are many relevant single-item settings where the single-crossing condition does not hold. For example, suppose that the signals indicate demand for a product being auctioned, agents represent firms, and one firm has a stronger signal about demand but is in a weaker position to take advantage of that demand. A setting like this could yield valuations that do not satisfy the single-crossing condition. For a concrete example, consider the following scenario given by Dasgupta and Maskin [15] and Maskin [28].

**Example 1.** Suppose that oil can be sold in the market at a price of four dollars per unit and that two firms are competing for the right to drill for oil. Firm 1 has a fixed cost of one to produce oil and a marginal cost of two for each additional unit produced, whereas firm 2 has a fixed cost of two and a marginal cost of one for each additional unit produced. In addition, suppose that firm 1 does a private test and discovers that the expected size of the oil reserve is $s_1$ units. Then, $v_1(s_1, s_2) = (4-2)s_1 - 1 = 2s_1 - 1$, whereas $v_2(s_1, s_2) = (4-1)s_1 - 2 = 3s_1 - 2$. These valuations do not satisfy the single-crossing condition because firm 1 needs to win when $s_1$ is low and lose when $s_1$ is high.

In this paper, we step away from both the single-crossing assumption and simple single-parameter settings. We present the first welfare approximation guarantees for interdependent values in much more general settings, encompassing a wide range of combinatorial auctions. The generalized Vickrey auction (Ausubel [4]) for interdependent values is a generalization of the seminal Vickrey [40] auction designed for independent private values. Similarly, our auction for interdependent values can be viewed as an extension of the VCG auction (Clarke [12], Groves [21]) for independent private values.

Although the VCG auction maximizes welfare when valuations are private and independent and although the generalized Vickrey auction maximizes welfare when single crossing holds, maximizing social welfare in combinatorial interdependent settings is known to be impossible (Dasgupta and Maskin [15], Jehiel and Moldovanu [23]). We circumvent this impossibility result by relaxing the goal of welfare maximization and considering approximation guarantees. In very general combinatorial settings and subject to a very natural condition (which we term "submodularity over signals"(SOS)), we devise an incentive-compatible mechanism that is guaranteed to give at least 1/4 of the optimal social welfare.

### 1.2. Research Problems

This paper addresses the following two issues related to social welfare maximization in the interdependent values model.

1. To what extent can the optimal social welfare be approximated in interdependent settings that do not satisfy the single-crossing condition?
2. How far beyond the single-item, single-dimensional setting can we go?

Given the impossibility result of Jehiel and Moldovanu [23], we ask if it is possible to *approximately* maximize social welfare in *combinatorial auctions with interdependent values*.

The first question was recently considered by Eden et al. [19], who gave two examples pointing out the difficulty of approximating social welfare without single crossing. Example 2 shows that even with two bidders and one signal, there are valuation functions for which no deterministic auction can achieve *any* bounded approximation ratio to optimal social welfare.

**Example 2** (No Bound for Deterministic Auctions (Eden et al. [19])). A single item is for sale. There are two players, $A$ and $B$; only $A$ has a signal $s_A \in \{0,1\}$. The valuations are

$$\begin{array}{ll} v_A(0) = 1 & v_B(0) = 0 \\ v_A(1) = 2 & v_B(1) = H, \end{array}$$

where $H$ is an arbitrary large number. If $A$ does not win when $s_A = 0$, then the approximation ratio is infinite. On the other hand, if $A$ does win when $s_A = 0$, then by monotonicity, $A$ must also win at $s_A = 1$, yielding a $2/H$ fraction of the optimal social welfare.

The next example can be used to show that there are valuation functions for which no randomized auction performs better (in the worst case) than allocating to a random bidder (i.e., a factor $n$ approximation to social welfare), even if a prior over the signals is known.

**Example 3** ($n$ Lower Bound for Randomized Auctions (Eden et al. [19])). There are $n$ bidders $1, \ldots, n$ that compete over a single item. For every agent $i$, $s_i \in \{0,1\}$, and

$$v_i(\mathbf{s}) = \prod_{j \neq i} s_j + \epsilon \cdot s_i \quad \text{for } \epsilon \to 0;$$

that is, agent $i$'s value is high if and only if all other agents' signals are high simultaneously. When all signals are one, then in any feasible allocation, there must be an agent $i$, which is allocated with probability of at most $1/n$. By monotonicity, this means that the probability this agent is allocated when the signal profile is $\mathbf{s}' = (\mathbf{1}_{-i}, 0_i)$ is at most $1/n$ as well. Therefore, the achieved welfare at signal profile $\mathbf{s}'$ is at most $1/n + (n-1) \cdot \epsilon$, whereas the optimal welfare is one, giving a factor $n$ gap. Eden et al. [19] show that there exists a prior for which the $n$ gap still holds, *even* if the mechanism knows the prior.

Therefore, *some* assumption is needed if we are to get good approximations to social welfare. The approach taken by Eden et al. [19] was to define a relaxed notion of single crossing that they called $c$-single crossing and then, provide mechanisms that approximately maximize social welfare, where the approximation ratio depends on $c$ and $n$, the number of agents.

In this paper, we go in a different direction starting with the observation that in Example 3, the valuations treat the signals as highly complementary; one has a value bounded away from zero only if all other agent's signals are high simultaneously. This suggests that the case where the valuations treat the signals more like "substitutes" might be easier to handle.

We capture this by focusing on SOS valuations. This means that for every $i$ and $j$, when signals $\mathbf{s}_{-j}$ are lower, the sensitivity of the valuation $v_i(\mathbf{s})$ to changes in $s_j$ is higher. Formally, we assume that for all $j$, for any $s_j$, $\delta \geq 0$, and for any $\mathbf{s}_{-j}$ and $\mathbf{s}'_{-j}$ such that component wise, $\mathbf{s}_{-j} \leq \mathbf{s}'_{-j}$, it holds that

$$v_i(s_j + \delta, \mathbf{s}_{-j}) - v_i(s_j, \mathbf{s}_{-j}) \geq v_i(s_j + \delta, \mathbf{s}'_{-j}) - v_i(s_j, \mathbf{s}'_{-j}).$$

Many valuations considered in the literature on interdependent valuations are SOS (although this term is not used) (Dasgupta and Maskin [15], Klemperer [26], Milgrom and Weber [30]). The simplest (yet still rich) class of SOS valuations is *fully separable* valuation functions, where there are *arbitrary* (weakly increasing) functions $g_{ij}(s_j)$ for each pair of bidders $i$ and $j$ such that

$$v_i(\mathbf{s}) = \sum_{j=1}^{n} g_{ij}(s_j).$$

This type of valuation function is ubiquitous in the economics literature on interdependent settings, often with the function simply assumed to be a linear function of the signals (see, e.g., Jehiel and Moldovanu [23], Klemperer [26]).

The family of fully separable functions includes the weighted sum valuations discussed and the mineral rights model (Roughgarden and Talgam-Cohen [37], Wilson [42]). A more general class of SOS valuation functions includes functions of the form $v_i(\mathbf{s}) = f(\sum_{j=1}^{n} g_{ij}(s_j))$, where $f$ is a weakly increasing concave function.

We can now state the main question we study in this paper. *To what extent can social welfare be approximated in interdependent settings with SOS valuations*? Unfortunately, Example 2 itself describes SOS valuations, so no deterministic auction can achieve any bounded approximation ratio, even for this subclass of valuations. Thus, we must turn to randomized auctions.

### 1.3. Our Results and Techniques

All of our positive results concern the design of randomized, prior-free, universally ex post IC, individually rational (IR) mechanisms. Prior free means that the rules of the mechanism make no use of the prior distribution over the signals and thus, need not have any knowledge of the prior.

Our first result provides approximation guarantees for single-parameter downward-closed settings. An important special case of this result is single-item auctions, which was the focus of Eden et al. [19].

See Theorem 1 (in Section 4). For every single-parameter downward-closed setting, if the valuation functions are SOS, then the `random sampling Vickrey (RS-V)` auction is a universally ex post IC-IR mechanism that gives a four approximation to the optimal social welfare.

Interestingly, no deterministic mechanism can give better than an $(n-1)$ approximation for arbitrary downward-closed settings, even if the valuations are single crossing, and this is tight. Recall that for a single-item auction or even multiple identical items, with single-crossing valuations, the deterministic generalized Vickrey auction obtains the optimal welfare (Ausubel [4], Maskin [28]).

We then turn to multidimensional settings. In the most general combinatorial auction model that we consider, each agent $i$ has a signal $s_{iT}$ for each subset $T$ of items and a valuation function $v_{iT} := v_{iT}(s_{1T}, s_{2T}, \ldots, s_{nT})$. For this setting, maximizing social welfare in ex post equilibrium might be impossible (see related work and also Propositions A.1, A.2, and B.2, which show that under one natural generalization of single crossing for combinatorial settings, single crossing is not sufficient for full efficiency).

However, rather surprisingly, for the case of *separable SOS* valuations (see Definition 16), we are able to extend the four-approximation guarantee to combinatorial auctions. Such valuations are prevalent in the literature and generalize the fully separable case discussed.

See Theorem 3 (in Section 5). For every combinatorial auction, if the valuation functions are separable SOS, then the `random sampling VickreyClarkeGrove (RS-VCG)` auction is a universally ex post IC-IR mechanism that gives a four approximation to the optimal social welfare.

Finally, we consider combinatorial auctions where each agent $i$ has a single-dimensional signal $s_i$ but where the valuation function $v_{iT}$ for each subset of items $T$ is an *arbitrary* SOS valuation function $v_{iT}(s_1, \ldots, s_n)$. For this case, we show the following.

See Theorems 4 and 5 (in Sections 6.1 and 6.2). Consider combinatorial auctions with single-dimensional signals, where each signal takes one of $k$ possible values. If the valuation functions are SOS, then there exists a universally ex post IC-IR mechanism that gives a $(k+3)$ approximation to the optimal social welfare. If the valuations are strong SOS (see Definition 14), the approximation ratio improves to $O(\log k)$.

All of the results, as well as our lower bounds, are summarized in Table 1. In addition, all of the results in this paper generalize easily, with a corresponding degradation in the approximation ratio, to the weaker requirement of a *d-approximate submodular over signals* (*d-SOS*) valuations (see Definition 13).

#### 1.3.1. Intuition for Results.

The fundamental tension in settings with interdependent valuations that is not present in the private values setting is the following. Consider, for example, a single-item auction setting where agent 1's truthful report of her signal increases agent 2's *value*. Because this increases the chance that agent 2 wins and may decrease agent 1's chance of winning, it might motivate agent 1 to strategize and misreport.

Our approach is to simply *prevent* this interaction. Without looking at the signals, our mechanism randomly divides the agents into two sets: potential winners and certain losers. Losers never receive any allocation. When estimating the

**Table 1.** The approximation factors achievable for social welfare maximization with SOS and strong-SOS valuations. Similar results hold for $d$-approximate SOS/strong-SOS valuations while losing a factor that depends on $d$. All positive results are obtained with universally ex post IC-IR randomized mechanisms.

| Setting | Approximation guarantees |
|---|---|
| Single-parameter SOS valuations, downward-closed feasibility, single-dimensional signals | $\leq 4$, $\forall$mech. $\geq 2$ (Section 4) |
| Arbitrary combinatorial SOS valuations, single-dimensional signals, $k$-sized signal space | $\leq k + 3$, $\forall$mech. $\geq 2$ (Section 6.1) |
| Arbitrary combinatorial strong-SOS valuations, single-dimensional signals, $k$-sized signal space | $\leq \log(k) + 2$, $\forall$mech. $\geq 2$ (Section 6.2) |
| Combinatorial separable-SOS valuations, multidimensional signals | $\leq 4$, $\forall$mech. $\geq 2$ (Section 5) |

value of a potentially winning agent $i$, we use only the signals of losers and $i$'s own signal(s). Thus, potential winners cannot impact the estimated values and hence, allocations of other potential winners. This resolves the truthfulness issue. The remaining question is as follows. Can we get sufficiently accurate estimates of the agents' values when we ignore so many signals?

The key lemma (Lemma 2 in Section 3) shows that we can do so when the valuations are SOS. Specifically, for any agent $i$, if all agents other than $i$ are split into two random sets $A$ (losers) and $B$ (potential winners) and the signals of agents in the random subset $B$ are "zeroed out," then the expected value agent $i$ has for the item is at least half of her true valuation. That is,

$$E_A[v_i(s_i, \mathbf{s}_A, \mathbf{0}_B)] \geq \frac{1}{2} v_i(\mathbf{s}).$$

Dealing with combinatorial settings is more involved as the truthfulness characterization is less obvious, but the key ideas of random partitioning and using the signals of certain losers remain at the core of our results.

**1.3.2. Additional Remarks.** Although this paper deals entirely with welfare maximization, our results have significance for the objective of maximizing the seller's revenue. Eden et al. [19] give a reduction from revenue maximization to welfare maximization in single-item auctions under a weaker submodularity condition, which follows from SOS. Thus, the constant factor approximation mechanism presented in this paper implies a constant factor approximation to the optimal revenue in single-item auctions with SOS valuations. We note that this is the first revenue approximation result that does not assume any single crossing-type assumption (Chawla et al. [10], Eden et al. [19], Li [27], and Roughgarden and Talgam-Cohen [37] require single crossing or approximate single crossing).

Finally, one can easily verify that, based on Yao's min-max theorem, the existence of a *randomized prior-free mechanism* that gives some approximation guarantee (in expectation over the coin flips of the mechanism) implies the existence of a *deterministic prior-dependent* mechanism that gives the same approximation guarantee (in expectation over the signal profiles).

### 1.4. More on Related Work

As discussed, in single-parameter settings, there is an extensive literature on mechanism design with interdependent valuations that considers social welfare maximization, revenue maximization, and other objectives. However, the vast majority of this literature assumes some kind of single-crossing condition and in the context of social welfare, focuses on exact optimization.

There are two papers that we are aware of that study the question of how well optimal social welfare can be approximated in ex post equilibrium without single crossing. The first is the aforementioned paper (Eden et al. [19]) on single-item auctions with interdependent valuations. They defined a parameterized version of single crossing, termed $c$-single crossing, where $c > 1$ is a parameter that indicates how close the valuation profile is to satisfy single crossing. For $c$–single-crossing valuations, they provide a number of results, including a lower bound of $c$ on the approximation ratio achievable by any mechanism, a matching upper bound for binary signal spaces, and mechanisms that achieve approximation ratios of $(n-1)c$ and $2c^{3/2}\sqrt{n}$ (the first is deterministic, and the second is randomized).

Ito and Parkes [22] also consider approximating social welfare in the interdependent setting. Specifically, they propose a greedy contingent-bid auction (a la Dasgupta and Maskin [15]) and show that it achieves a $\sqrt{m}$ approximation to the optimal social welfare for $m$ goods, in the special case of combinatorial auctions with single-minded bidders.

For multidimensional signals and settings, the landscape is sparser (and bleaker) and to our knowledge, focuses on exact social welfare maximization. Maskin [28] has observed that, in general, no efficient incentive-compatible single-item auction exists if a buyer's valuation depends on a multidimensional signal.

Jehiel and Moldovanu [23] consider a very general model in which there is a set $K$ of possible alternatives and a multidimensional signal space, where each agent $j$ has a signal $s^j_{ki}$ for each outcome $k$ and other agent $j$. In their model, the valuation function of an agent $i$ for outcome $k$ is linear in the signals: that is, $v_i(k) := \sum_j a^j_{ki} s^j_{ki}$. Thus, their valuation functions are, in one sense, a special case of our separable valuation functions. On the other hand, they are more general in that all quantities depend on the outcome $k$. Thus, there are allocation externalities. Their main result is that, generically, there is no Bayes–Nash incentive-compatible mechanism that maximizes social welfare in this setting. However, they do give an ex post IC mechanism that maximizes social welfare with both information and allocation externalities if the signals are one-dimensional, the valuation functions are linear in the signals, and a single crossing-type condition holds.

Jehiel et al. [24] go on to show that the only deterministic social choice functions that are ex post implementable in generic mechanism design frameworks with multidimensional signals, interdependent valuations and transferable utilities, are constant functions.

Finally, Bikhchandani [9] considers a single-item setting with multidimensional signals but no allocation externalities and shows that there is a generalization of single crossing that allows some social choice rules to be implemented ex post.

For further analysis and discussion of implementation with interdependent valuations, see, for example, Bergemann and Morris [6] and McLean and Postlewaite [29].

For further literature in computer science on interdependent and correlated values, see Abraham et al. [1], Babaioff et al. [5], Chawla et al. [10], Che et al. [11], Constantin and Parkes [13], Constantin et al. [14], Dobzinski et al. [17], Syrgkanis et al. [39], Klein et al. [25], Li [27], Papadimitriou and Pierrakos [33], Robu et al. [34], and Ronen [36].

The idea of using a random sampling approach in the mechanism design literature was introduced in Goldberg et al. [20].

## 2. Model and Definitions

### 2.1. Single-Parameter Settings

In Section 4, we will consider single-parameter settings with interdependent valuations and downward-closed feasibility constraints. In these settings, a mechanism decides which subsets of agents $1, \ldots, n$ are to receive "service" (e.g., an item). The feasibility constraint is defined by a collection $\mathcal{I} \subseteq 2^{[n]}$ of subsets of agents that may feasibly be served simultaneously. We restrict attention to *downward-closed settings*, which means that any subset of a feasible set is also feasible. A simple example is a $k$-item auction, where $\mathcal{I}$ is the collection of all subsets of agents of size at most $k$. For these settings, we use the interdependent value model of Milgrom and Weber [30].

**Definition 1** (Single-Dimensional Signals, Single-Parameter Valuations). Each agent $j$ has a private signal $s_j \in \mathbb{R}^+$. The value agent $j$ gives to "receiving service" $v_j(\mathbf{s}) \in \mathbb{R}^+$ is a function of all agents' signals $\mathbf{s} = (s_1, s_2, \ldots, s_n)$. The function $v_j(\mathbf{s})$ is assumed to be weakly increasing in each coordinate and strictly increasing in $s_i$.

#### 2.1.1. Deterministic Mechanisms.

We provide formal definitions of the key concepts used pertaining to deterministic mechanisms.

**Definition 2** (Deterministic Single-Parameter Mechanisms). A deterministic mechanism $M = (x, p)$ in the downward-closed setting is a mapping from reported signals $\mathbf{s} = (s_1, \ldots, s_n)$ to allocations $x(\mathbf{s}) = \{x_i(\mathbf{s})\}_{1 \le i \le n}$ and payments $p(\mathbf{s}) = \{p_i(\mathbf{s})\}_{1 \le i \le n}$, where $x_i(\mathbf{s}) \in \{0, 1\}$ indicates whether agent $i$ receives service and $p_i(\mathbf{s})$ is the payment of agent $i$. It is required that the set of agents that receive service is feasible (i.e., $\{i | x_i(\mathbf{s}) = 1\} \in \mathcal{I}$). (The mechanism designer knows the form of the valuation functions but learns the private signals only when they are reported.)

In order to reason about an agent's behavior within a mechanism, we must define agent utility.

**Definition 3** (Agent Utility). Given a deterministic mechanism $(x, p)$, the *utility* of agent $i$ when her true signal is $s_i$, she reports $s_i'$, and the other agents report $\mathbf{s}_{-i}$ is

$$u_i(s_i', \mathbf{s}_{-i} | s_i) = x_i(s_i', \mathbf{s}_{-i}) v_i(s_i, \mathbf{s}_{-i}) - p_i(s_i', \mathbf{s}_{-i}).$$

Agent $i$ will report $s_i'$ so as to maximize $u_i(s_i', \mathbf{s}_{-i} | s_i)$. We use $u_i(\mathbf{s})$ to denote the utility when she reports truthfully (i.e., $u_i(s_i, \mathbf{s}_{-i} | s_i)$).

Given this definition of utility, we can now define our notion of truthfulness.

**Definition 4** (Deterministic Ex Post IC). A deterministic mechanism $M = (x, p)$ in the interdependent setting is *ex post* IC if, irrespective of the true signals and given that all other agents report their true signals, there is no advantage to an agent to report any signal other than her true signal. In other words, assuming that $\mathbf{s}_{-i}$ are the true signals of other bidders, $u_i(s_i', \mathbf{s}_{-i} | s_i)$ is maximized by reporting $s_i$ truthfully.

Another natural condition for our agents to want to participate and tell the truth is ex post individual rationality.

**Definition 5** (Deterministic Ex Post IR). A deterministic mechanism in the interdependent setting is *ex post* IR if, irrespective of the true signals and given that all other agents report their true signals, no agent gets negative utility by participating in the mechanism.

If a deterministic mechanism is both ex post IC and ex post IR, we say that it is ex post IC-IR. In order to achieve these conditions in the single-parameter setting, we must have a monotone allocation rule.

**Definition 6.** A deterministic allocation rule $x$ is monotone if for every agent $i$, every signal profile of all other agents $s_{-i}$, and every $s_i \le s_i'$, it holds that $x_i(s_i, s_{-i}) = 1 \Rightarrow x_i(s_i', s_{-i}) = 1$.

**Proposition 1** (Roughgarden and Talgam-Cohen [37]). *For every deterministic allocation rule $x$ for single-parameter valuations, there exist payments $p$ such that the mechanism $(x, p)$ is ex post IC-IR if and only if $x_i$ is monotone for every agent $i$.*

#### 2.1.2. Randomized Mechanisms.
Now, we add on the concepts relating to randomized mechanisms.

**Definition 7.** A randomized mechanism is a probability distribution over deterministic mechanisms.

We need a slightly different notion of truthfulness when our mechanism is randomized.

**Definition 8** (Universal Ex Post IC-IR). A randomized mechanism is said to be universally ex post IC-IR if all deterministic mechanisms in the support are ex post IC-IR.

### 2.2. Combinatorial Valuations with Interdependent Signals

Sections 5 and 6 focus on combinatorial auctions, where there are $n$ agents and $m$ items. In these settings, a mechanism is used to decide how the items are partitioned among the agents. We consider two models for the interdependent valuations.

**Definition 9** (Single-Dimensional Signals, Combinatorial Valuations). Each agent $i$ has a signal $s_i \in \mathbb{R}^+$. The value agent $i$ gives to the subset of items $T \subseteq [m]$, which we denote by $v_{jT}(\mathbf{s})$, is a function of $\mathbf{s} = (s_1, s_2, \ldots, s_n)$.

In this first model, each agent $i$ only has a single piece of information, $s_i$, and these $n$ aggregate pieces of information (one per agent) are relevant to any combinatorial set of items $T$. Hence, there is a valuation function for each agent $i$ and the relevant set of items $T$, and it depends on these $n$ pieces of information. In the second model, the agents rather have a ($n$-independent) piece of information that is relevant for each separate set of items, so $i$'s value for the set $T$ depends on the information from all $n$ agents for the bundle $T$ specifically—the signals from agent 1 $s_{1T}$, agent 2 $s_{2T}$, and so forth.

**Definition 10** (Multidimensional Combinatorial Signals, Combinatorial Valuations). Here, each agent has a signal for each subset of items; for any agent $i$, we use $s_{iT}$ to denote agent $i$'s signal for subset of items $T \subseteq [m]$. The value agent $i$ gives to set $T$ is denoted by $v_{iT}(\mathbf{s}_T)$, where $\mathbf{s}_T = (s_{1T}, s_{2T}, \ldots, s_{nT}) \in \mathbb{R}^{+n}$. We use $\mathbf{s}$ to denote the set of all signals $\{\mathbf{s}_T\}_{T \subseteq 2^m}$.

In both cases, each $v_{iT}(\cdot)$ is assumed to be a weakly increasing function of each signal and strictly increasing in $s_i$ (or $s_{iT}$, respectively) and known to the mechanism designer.

We give subsequent definitions only for multidimensional combinatorial signals, as single-dimensional signals can be viewed as a special case of multidimensional signals where $s_{iT} = s_i$ for all $T$.

#### 2.2.1. Deterministic Mechanisms

**Definition 11** (Deterministic Mechanisms for Combinatorial Settings). A deterministic mechanism $M = (x, p)$ is a mapping from reported signals $\mathbf{s}$ to allocations $x = \{x_{iT}\}$ (where each $x_{iT} \in \{0, 1\}$) and payments $p = \{p_{iT}\}$ for all $1 \le i \le n$ and $T \subset \{1, \ldots, m\}$.

- Agent $j$ is allocated the set $T$ if and only if $x_{jT}(\mathbf{s}) = 1$.
- For each agent $j$, there is at most one $T$ for which $x_{jT}(\mathbf{s}) = 1$.
- The sets allocated to different agents do not intersect.
- The payment for agent $j$ when her allocation is set $T$ is $p_{jT}(\mathbf{s})$.

**Definition 12** (Agent Utility). The *utility* of agent $i$ when her signals are $\mathbf{s}_i = \{s_{iT}\}_{T \subset 2^m}$, she reports $\mathbf{s}'_i$, and the other agents report $\mathbf{s}_{-i}$ is

$$u_i(\mathbf{s}'_i, \mathbf{s}_{-i} | \mathbf{s}_i) = \sum_{T \subseteq 2^m} x_{iT}(\mathbf{s}'_i, \mathbf{s}_{-i})[v_{iT}(\mathbf{s}_{iT}, \mathbf{s}_{-iT}) - p_{iT}(\mathbf{s}'_i, \mathbf{s}_{-i})].$$

Given a mechanism $M = (x, p)$, agent $i$ will report $\mathbf{s}'_i$ so as to maximize $u_i(\mathbf{s}'_i, \mathbf{s}_{-i} | \mathbf{s}_i)$. We use $u_i(\mathbf{s})$ to denote the utility when she reports truthfully (i.e., $u_i(\mathbf{s}_i, \mathbf{s}_{-i} | \mathbf{s}_i)$).

The definitions of *ex post* IC and *ex post* IR for deterministic mechanisms for combinatorial settings are the same as the appropriate definitions for single-parameter mechanisms (Definitions 4 and 5 with the obvious modifications).

#### 2.2.2. Randomized Mechanisms.
As with single-parameter mechanisms, a randomized mechanism for a combinatorial setting is a probability distribution over deterministic mechanisms for the combinatorial setting, and a randomized mechanism is said to be *universally ex post IC-IR* if all deterministic mechanisms in the support are themselves ex post IC-IR.

### 2.3. SOS

As discussed in Section 1, our results will rely on an assumption about the valuation functions that we call SOS. The SOS (strong-SOS) notion we use is the same as the weak diminishing returns (strong diminishing returns) submodularity notion in Bian et al. [8] and Niazadeh et al. [32] (weak diminishing returns submodularity was introduced in Soma and Yoshida [38], where it is termed "diminishing returns submodularity"). SOS was also used in Eden et al. [19], generalizing a similar notion in Chawla et al. [10].

**Definition 13** (*d*-Approximate Submodular over Signals Valuations). A valuation function $v(\mathbf{s})$ is a *d-SOS valuation* if for all $j$, $s_j$, $\delta \geq 0$,

$$\mathbf{s}_{-j} = (s_1, \ldots, s_{j-1}, s_{j+1}, \ldots, s_n) \quad \text{and} \quad \mathbf{s}'_{-j} = (s'_1, \ldots, s'_{j-1}, s'_{j+1}, \ldots, s'_n),$$

such that $\mathbf{s}'_{-j}$ is smaller than or equal to $\mathbf{s}_{-j}$ coordinate wise; it holds that

$$d \cdot (v(\mathbf{s}'_{-j}, s_j + \delta) - v(\mathbf{s}'_{-j}, s_j)) \geq v(\mathbf{s}_{-j}, s_j + \delta) - v(\mathbf{s}_{-j}, s_j). \tag{1}$$

If $v$ satisfies this condition with $d = 1$, we say that $v$ is an SOS valuation function.

Essentially, when agents have SOS valuations, they are more sensitive to a change in some agent $j$'s signal—or agent $j$ acquiring new information—when everyone else's signals are *lower*: that is, when the other agents have *less* information. On the other hand, we also consider the notion of *strong-SOS* valuations. The difference here is that in addition to being more sensitive when others have lower signals, agents are also more sensitive to agent $j$'s signal changing—agent $j$ acquiring new information—when agent $j$'s signal started lower (the agent started with less information). That is, in addition to submodularity, the valuation functions are essentially concave in $s_j$, where increasing in $s_j$ also has a diminishing effect. For an example, consider the following additive functions, which are trivially SOS; $v(s_1, s_2) = s_1 + s_2$ is also strong SOS, whereas $v(s_1, s_2) = 2^{s_1} + 2^{s_2}$ only satisfies SOS.

**Definition 14** (*d*-Approximate Strong Submodular over Signals Valuations). A valuation function $v(\mathbf{s})$ is a *d-approximate strong submodular over signals* (*d-strong-SOS*) *valuation* if for any $j$, $\delta \geq 0$,

$$\mathbf{s} = (s_1, \ldots, s_n) \quad \text{and} \quad \mathbf{s}' = (s'_1, \ldots, s'_n),$$

such that $\mathbf{s}'$ is smaller than or equal to $\mathbf{s}$ coordinate wise; it holds that

$$d \cdot (v(\mathbf{s}'_{-j}, s'_j + \delta) - v(\mathbf{s}'_{-j}, s'_j)) \geq v(\mathbf{s}_{-j}, s_j + \delta) - v(\mathbf{s}_{-j}, s_j). \tag{2}$$

If $v$ satisfies this condition with $d = 1$, we say that $i$'s valuation functions are "strong SOS."

**Definition 15** (SOS Valuations Settings). We say that a mechanism design setting with interdependent valuations is an *SOS valuations setting* or equivalently, that the agents have SOS valuations in each of the following cases.

- Single-parameter valuations (as in Definition 1). For every $i$, the valuation function $v_i(\mathbf{s})$ is SOS.
- Combinatorial valuations with single-parameter signals (as in Definition 9). For every $i$ and $T$, the valuation function $v_{iT}(\mathbf{s})$ is SOS.
- Combinatorial valuations with multiparameter signals (as in Definition 10). For every $i$ and $T$, $v_{iT}(\mathbf{s}_T)$ is SOS, where $\mathbf{s}_T = (s_{1T}, \ldots, s_{nT})$.

Similar definitions can be given for $d$-SOS valuation settings and $d$-strong-SOS valuation settings.

Finally, in Section 5, we will specialize to the case of *separable* SOS valuations.

**Definition 16** (Separable SOS Valuations). We say that sets of valuations as in Definition 10 are *separable SOS valuations* if for every agent $i$ and subset $T$ of items, $v_{iT}(\mathbf{s}_T)$ can be written as

$$v_{iT}(\mathbf{s}_T) = g_{-iT}(\mathbf{s}_{-iT}) + h_{iT}(s_{iT}),$$

where $g_{-iT}(\cdot)$ is weakly increasing, $h_{iT}(\cdot)$ is strictly increasing, and $g_{-iT}(\mathbf{s}_{-iT})$ is itself an SOS valuation function.

**Observation 1.** A separable SOS valuation function is itself an SOS valuation function.

We can similarly define separable $d$-SOS valuations.

### 2.4. Comparison of Various Valuation Classes

Previous studies primarily use the SC condition in order to obtain positive results. Our work concentrates on the SOS condition, which enables stronger results. Nevertheless, the SC and SOS conditions are noncomparable, not even with respect to their approximate notions. In particular, the valuations in Example 2 satisfy SOS but not single crossing and require an arbitrarily big $c$ to satisfy $c$-SC. Similarly, the valuations in the example in Eden et al. [18, proposition 2.3] are SC but not SOS and not even $d$-SOS for any $d$.

Example 2 shows that under public valuations, deterministic mechanisms (which have been the focus in the IDV literature until recently) cannot give any approximation in the absence of SC, even under the strong special case of SOS of fully separable valuations. The latter example (from Eden et al. [18]) shows that under private valuations, no mechanism (deterministic or randomized) can give better than $n$ approximation in the absence of SOS, even if SC holds.

### 2.5. A Useful Fact About SOS Valuations

**Lemma 1** (Submodularity over Sets of Signals). *Let $v:\mathbb{R}^{+n}\to\mathbb{R}^+$ be a d-SOS function. Let $A\subseteq[n]$ and $B=[n]\setminus A$. For any $\mathbf{s}_A,\mathbf{y}_A\in\mathbb{R}^{+|A|}$, and $\mathbf{s}_B,\mathbf{s}'_B\in\mathbb{R}^{+|B|}$ such that $\mathbf{s}_B$ is smaller than $\mathbf{s}'_B$ coordinate wise,*

$$d\cdot(v(\mathbf{s}_A+\mathbf{y}_A,\mathbf{s}_B)-v(\mathbf{s}_A,\mathbf{s}_B))\geq v(\mathbf{s}_A+\mathbf{y}_A,\mathbf{s}'_B)-v(\mathbf{s}_A,\mathbf{s}'_B).$$

**Proof.** Let $i_1,i_2,\ldots,i_{|A|}$ be the elements of $A$. For $1\leq j\leq|A|$, let $\mathbf{s}^j$ and $\mathbf{s}'^j$ denote the vectors

$$\mathbf{s}^j=((s_{i_1}+y_{i_1}),\ldots,(s_{i_j}+y_{i_j}),s_{i_{j+1}},\ldots,s_{i_{|A|}},\mathbf{s}_B),$$
$$\mathbf{s}'^j=((s_{i_1}+y_{i_1}),\ldots,(s_{i_j}+y_{i_j}),s_{i_{j+1}},\ldots,s_{i_{|A|}},\mathbf{s}'_B).$$

Note that $\mathbf{s}^{|A|}=(\mathbf{s}_A+\mathbf{y}_A,\mathbf{s}_B)$ and $\mathbf{s}'^{|A|}=(\mathbf{s}_A+\mathbf{y}_A,\mathbf{s}'_B)$.

It follows from the $d$-SOS definition that for every $1\leq j\leq|A|$,

$$d\cdot(v(\mathbf{s}^j)-v(\mathbf{s}^{j-1}))\geq v(\mathbf{s}'^j)-v(\mathbf{s}'^{j-1}),\tag{3}$$

where $\mathbf{s}^0=(\mathbf{s}_A,\mathbf{s}_B)$ and $\mathbf{s}'^0=(\mathbf{s}_A,\mathbf{s}'_B)$.

Summing Equation (3) for $j=1,2,\ldots,|A|$ proves the claim. □

## 3. The Key Lemma

The following is a key lemma, which is used for both single-parameter and combinatorial settings.

**Lemma 2.** *Let $v_i:\mathbb{R}^{+n}\to\mathbb{R}^+$ be a d-SOS function. Let $A$ be a uniformly random subset of $[n]\setminus\{i\}$, and let $B:=([n]\setminus\{i\})\setminus A$. It now holds that*

$$\mathbb{E}_A[v_i(\mathbf{s}_A,\mathbf{0}_B,s_i)]\geq\frac{1}{d+1}v_i(\mathbf{s}),$$

*where the expectation is over the random choice of $A$.*

**Proof.** We consider two equiprobable events.

- Event 1: $A=S\subset[n]\setminus\{i\}$ is chosen as the random subset, and
- Event 2: $A=T=([n]\setminus\{i\})\setminus S$ is chosen as the random subset.

Normalize the valuations so that $v_i(\mathbf{s})=1$, and define $\alpha,\beta\in[0,1]$ such that

$$v_i(\mathbf{s}_S,\mathbf{0}_T,s_i)=\alpha,\qquad v_i(\mathbf{0}_S,\mathbf{s}_T,s_i)=\beta.$$

It follows that

$$\begin{aligned}\beta=v_i(\mathbf{0}_S,\mathbf{s}_T,s_i)&\geq v_i(\mathbf{0}_S,\mathbf{s}_T,s_i)-v_i(\mathbf{0}_S,\mathbf{0}_T,s_i)\\&\geq(v_i(\mathbf{s}_S,\mathbf{s}_T,s_i)-v_i(\mathbf{s}_S,\mathbf{0}_T,s_i))/d\\&=(1-\alpha)/d\\&\iff\\ \beta d+\alpha\geq 1,&\end{aligned}$$

where the first inequality follows from nonnegativity of $v_i(\mathbf{0}_S,\mathbf{0}_T,s_i)$, and the second inequality follows from $v_i$ being $d$-SOS and Lemma 1.

Similarly, we have that

$$\alpha d+\beta\geq 1.$$

Adding these two inequalities and rearranging gives

$$\alpha+\beta\geq\frac{2}{d+1}.$$

Partition the event space into pairs $(S,T)$ that partition $[n]\setminus\{i\}$. For every such $(S,T)$ pair, it follows that $v_i(\mathbf{s}_S,\mathbf{0}_T,s_i)+v_i(\mathbf{0}_S,\mathbf{s}_T,s_i)=\alpha+\beta\geq 2/(d+1)$.

We conclude with the following, where the third line follows from the fact that there are $2^{n-1}/2$ such $(S, T)$ pairs that partition $[n] \setminus \{i\}$:

$$\begin{aligned}\mathbb{E}_A[v_i(\mathbf{s}^A, \mathbf{0}_B, s_i)] &= \sum_{A \subseteq [n]\setminus\{i\}} \Pr[A] \cdot v_i(\mathbf{s}^A, \mathbf{0}_B, s_i) \\ &= \frac{1}{2^{n-1}} \cdot \sum_{A \subseteq [n]\setminus\{i\}} v_i(\mathbf{s}^A, \mathbf{0}_B, s_i) \\ &\geq \frac{1}{2^{n-1}} \cdot \frac{2^{n-1}}{2} \cdot \frac{2}{d+1} = \frac{1}{d+1},\end{aligned}$$

as desired. □

## 4. Single-Parameter Valuations

In this section, we describe the `RS-V mechanism` that achieves a four approximation for single-parameter downward-closed environments with SOS valuations and a $2(d+1)$ approximation for $d$-SOS valuations. We then give a lower bound of two and $\sqrt{d}$ for SOS and $d$-SOS valuations, respectively, even in the case of selling a single item.

Let $\mathcal{I} \subseteq 2^{[n]}$ be a downward-closed set system. We present a mechanism that serves only sets in $\mathcal{I}$ and gets a $2(d+1)$ approximation to the optimal welfare.

### 4.1. `RS-V`

- Elicit bids $\tilde{\mathbf{s}}$ from the agents.
- Partition the agents into two sets, $A$ and $B$, uniformly at random.
- For $i \in B$, let $w_i = v_i(\tilde{\mathbf{s}}_A, \tilde{s}_i, \mathbf{0}_{B\setminus\{i\}})$.
- Allocate to a set of bidders in

$$\arg\max_{S \in \mathcal{I}\,:\, S \subseteq B} \left\{ \sum_{i \in S} w_i. \right\}$$

**Theorem 1.** *For agents with SOS valuations and for every downward-closed feasibility constraint $\mathcal{I}$, `RS-V` is an ex post IC-IR mechanism that gives four approximation to the optimal welfare. For d-SOS valuations, the mechanism gives a $2(d+1)$ approximation to the optimal welfare.*

**Proof.** We first show the allocation is monotone in one's signal, and hence, by Proposition 1, the mechanism is ex post IC-IR. Fix a random partition $(A, B)$.

- Agents in $A$ are never allocated anything, and thus, their allocation is weakly monotone in their signal.
- For an agent $i \in B$, increasing $\tilde{s}_i$ can only increase $w_i$, whereas it leaves $w_j$ unchanged for all $j \in B \setminus \{i\}$. Thus, this only increases the weight of feasible sets (subsets of $B$ in $\mathcal{I}$) that $i$ belongs to. Therefore, increasing $s_i$ can only cause $i$ to go from being unallocated to being allocated.

For approximation, consider a set $S^* \in \arg\max_{S \in \mathcal{I}} \sum_{i \in S} v_i(\mathbf{s})$ that maximizes social welfare. For every $i \in S^*$, from Lemma 2, we have that

$$\mathbb{E}_B[w_i \cdot \mathbf{1}_{i \in B}] = \mathbb{E}_B[v_i(\mathbf{s}_i, \mathbf{s}_A, \mathbf{0}_{B_{-i}}) | i \in B] \cdot Pr(i \in B) \geq \frac{v_i(\mathbf{s})}{d+1} \cdot \frac{1}{2}. \tag{4}$$

For every set $B$, the fact that $\mathcal{I}$ is downward closed implies that $S^* \cap B \in \mathcal{I}$. Therefore, $S^* \cap B$ is eligible to be selected by `RS-V` as the allocated set of bidders. We have that the values of the bidders we allocate to are at least

$$\begin{aligned}\mathbb{E}_B\left[\max_{S \in \mathcal{I}: S \subseteq B} \sum_{i \in S} w_i\right] &\geq \mathbb{E}_B\left[\sum_{i \in S^* \cap B} w_i\right] = \mathbb{E}_B\left[\sum_{i \in S^*} w_i \cdot \mathbf{1}_{i \in B}\right] \\ &= \sum_{i \in S^*} \mathbb{E}_B[w_i \cdot \mathbf{1}_{i \in B}] \geq \sum_{i \in S^*} \frac{v_i(\mathbf{s})}{2(d+1)},\end{aligned}$$

as desired. Because the allocated bidders' true values at $\mathbf{s}$ are only higher than the proxy values $w_i$, this continues to hold. □

We note that for the case of downward-closed feasibility constraints, even if the valuations satisfy single crossing, there can be an $n-1$ gap between the optimal welfare and the welfare that the best deterministic mechanism can get. This is stated in Proposition B.1 in Appendix B.

The following lower bounds (Theorem 2) show that even for a single-item setting, one cannot hope to get a better approximation than two and $\Omega(\sqrt{d})$ for SOS and $d$-SOS valuations, respectively. These lower bounds apply to arbitrary randomized mechanisms.

**Theorem 2.** *No ex post IC-IR mechanism for selling a single item can get a better approximation than*

a. *a factor of two for SOS valuations and*
b. *a factor of $\Omega(\sqrt{d})$ for d-SOS valuations.*

**Proof.** Let $x_i(\mathbf{s})$ be the probability agent $i$ is allocated at signal profile $\mathbf{s}$. Notice that for every $\mathbf{s}$, $\sum_i x_i(\mathbf{s}) \le 1$; otherwise, the allocation rule is not feasible.

a. Consider the case where there are two agents, 1 and 2, $s_1 \in \{0, 1\}$, and agent 2 has no signal. The valuations are $v_1(0) = 1$, $v_1(1) = 1 + \epsilon$, $v_2(0) = 0$ and $v_2(1) = H$ for $H \gg 1 \gg \epsilon$. It is easy to see the valuations are SOS.
In order to get better than a two approximation at $s_1 = 0$, we must have $x_1(0) > 1/2$. By monotonicity, this forces $x_1(1) > 1/2$ as well, and hence, $x_2(1) < 1/2$ by feasibility. This implies that the expected welfare when $s_1 = 1$ is $x_1(1)v_1(1) + x_2(1)v_2(1) < H/2 + 1$, whereas the optimal welfare when $s_1 = 1$ is $H$. For a large $H$, this approaches a two approximation. Note that this lower bound applies even given a known prior distribution on the signals in the event that we have a prior on the signals that satisfies $\Pr[s_1 = 0] \cdot 1 = \Pr[s_1 = 1] \cdot H$.

b. Assume $d \ge 4$; otherwise, (a) gives an $\Omega(\sqrt{d})$ lower bound. Consider the case where there are $n = \sqrt{d}$ agents and $s_i \in \{0, 1\}$ for every agent $i$. The valuation of agent $i$ is

$$v_i(\mathbf{s}) = \begin{cases} \sum_{j \ne i} s_j + \epsilon \cdot s_i & \exists j \ne i : \ s_j = 0 \\ d + \epsilon \cdot s_i & s_j = 1 \ \ \forall j \ne i, \end{cases}$$

where $\epsilon \longrightarrow 0$.

To see that the valuations are $d$-SOS, notice that whenever a signal $s_j$ changes from zero to one, the valuation of agent $i \ne j$ increases by one *unless* all other signals beside $i$'s are already set to one, in which case the valuation increases by $d - \sqrt{d} + 2 < d$. Consider valuation profiles $\mathbf{s}^{\mathbf{i}} = (0_i, \mathbf{1}_{-i})$. Note that by monotonicity, for every truthful mechanism, it must be the case that $x_i(\mathbf{s}^{\mathbf{i}}) \le x_i(\mathbf{1})$. Because any feasible allocation rule must satisfy $\sum_{i=1}^{\sqrt{d}} x_i(\mathbf{1}) \le 1$, then it must be the case there exists some agent $i$ such that $x_i(\mathbf{1}) \le 1/\sqrt{d}$, which by monotonicity, implies that $x_i(\mathbf{s}^{\mathbf{i}}) \le 1/\sqrt{d}$. However, at profile $\mathbf{s}^{\mathbf{i}}$, $v_i(\mathbf{s}^{\mathbf{i}}) = d$, whereas $v_j(\mathbf{s}^{\mathbf{i}}) = \sqrt{d} - 2 < \sqrt{d}$ for all $j \ne i$; so, we get that the expected welfare of the mechanism at $\mathbf{s}^{\mathbf{i}}$ is at most $x_i(\mathbf{s}^{\mathbf{i}}) \cdot d + (1 - x_i(\mathbf{s}^{\mathbf{i}})) \cdot \sqrt{d} \le 2\sqrt{d}$, whereas the optimal welfare is $d$. Again, the lower bound also applies to the setting with known priors on the signals using a prior that satisfies $\Pr[\mathbf{s}^{\mathbf{i}}] = \Pr[\mathbf{s}^{\mathbf{j}}] = 1/\sqrt{d}$ for all $i$ and $j$. □

## 5. Combinatorial Auctions with Separable Valuations

In this section, we present an ex post IC-IR mechanism that gives 1/4 of the optimal social welfare in any combinatorial auction setting with separable SOS valuations (as in Definition 16). Recall that the valuation function of agent $i$ for a subset $T$ of items is separable SOS if it can be written as $v_{iT}(\mathbf{s}_T) = g_{-iT}(\mathbf{s}_{-iT}) + h_{iT}(\mathbf{s}_{iT})$, where $g_{-iT}$ is SOS. The mechanism that we call the `RS-VCG` auction is a natural extension of the `RS-V` auction presented in Section 4. Note that unlike `RS-V`, here we need to explicitly define payments so that the obtained mechanism is ex post IC-IR. We derive VCG-inspired payments, which align the objective of the mechanism with that of the agents. Separability is used here, as without it, the payment term would have been affected by the agent's report (whereas with separability, only the allocation is affected by it).

### 5.1. RS-VCG

- Agents report their signals $\tilde{\mathbf{s}}$.
- Partition the agents into two sets $A$ and $B$ uniformly at random.
- For each agent $j \in B$ and bundle $T \subseteq [m]$, let

$$w_{jT} := v_{jT}(\tilde{s}_{jT}, \tilde{\mathbf{s}}_{AT}, \mathbf{0}_{B_{-j}T}) = g_{-jT}(\tilde{\mathbf{s}}_{AT}, \mathbf{0}_{B_{-j}T}) + h_{jT}(\tilde{s}_{jT}).$$

- Let the allocation be

$$\{T_i\}_{i \in B} \in \arg \max_{\{S_i\}_{i \in B}} \sum_{i \in B} w_{iS_i};$$

that is, $\{T_i\}_{i \in B}$ is the allocation that maximizes the "welfare" using $w_{iT}$'s.

- Set the payment for a winning agent $i \in B$ receiving a set of goods $T_i$ to be

$$p_i(\tilde{\mathbf{s}}) := g_{-iT_i}(\tilde{\mathbf{s}}_{-iT_i}) - g_{-iT_i}(\tilde{\mathbf{s}}_{AT_i}, \mathbf{0}_{B_{-i}T_i}) - \sum_{j \in B \setminus \{i\}} w_{jT_j} + w_{-i},$$

where

$$w_{-i} = \max_{\text{partitions } \{T'_j\}} \sum_{j \in B \setminus \{i\}} w_{jT'_j},$$

and that is, $w_{-i}$ is the weight of the best allocation without agent $i$.

Because the $w_{jT}$'s do not depend on agent $i$'s report (because $i$ is in $B$), $w_{-i}$ does not depend on agent $i$'s report. Therefore, we can (and will) ignore this term when considering incentive compatibility.

Note also that the maximal partition guarantees that $w_{-i} \geq \sum_{j \in B \setminus \{i\}} w_{jT_j}$ and monotonicity of valuations in signals guarantees that $g_{-iT_i}(\tilde{\mathbf{s}}_{-i}) \geq g_{-iT_i}(\tilde{\mathbf{s}}_A, \mathbf{0}_{B_{-i}})$. Therefore, the payments $p_i(\tilde{\mathbf{s}})$ are always nonnegative.

**Theorem 3.** `Random sampling VCG` *is an ex post IC-IR mechanism that gives a four approximation to the optimal social welfare for any combinatorial auction setting with separable SOS valuations.*

**Proof.** First, we show that if the agents bid truthfully, then the mechanism gives a four approximation to social welfare. For every agent $i$ and bundle $T$,

$$\mathbb{E}_B[w_{iT} \cdot \mathbf{1}_{i \in B}] = \mathbb{E}_B[v_{iT}(\mathbf{s}_{iT}, \mathbf{s}_{AT}, \mathbf{0}_{B_{-i}T}) | i \in B] \cdot Pr(i \in B) \geq \frac{v_{iT}(\mathbf{s}_T)}{2} \cdot \frac{1}{2}, \tag{5}$$

where the inequality follows by applying Lemma 2 with $d = 1$.

Let $S_1^*, \ldots, S_n^*$ be the true welfare-maximizing allocation. Then,

$$\begin{aligned}\mathbb{E}_B\left[\max_{\text{partitions } \{T_i\}} \sum_{i \in B} w_{iT_i}\right] &\geq \mathbb{E}_B\left[\sum_i w_{iS_i^*} \cdot \mathbf{1}_{i \in B}\right] \\ &= \sum_i \mathbb{E}_B[w_{iS_i^*} \cdot \mathbf{1}_{i \in B}] \geq \frac{1}{4} \sum_i v_{iS_i^*}(\mathbf{s}_{S_i^*}),\end{aligned}$$

where the last inequality follows by substituting $S_i^*$ in $T$ in Equation (5) for every $i$. Because $v_{iT}(\mathbf{s})$ is always at least $w_{iT}$, this proves the approximation ratio.

Next, we show that `RS-VCG` is universally ex post IC. Fix a random partition $(A, B)$. Suppose that when all agents bid truthfully,

$$\{T_j^*\}_{j \in B} = \arg\max_{\text{partitions } \{T_j\}} \sum_{j \in B} w_{jT_j}.$$

Suppose that all agents but $i \in B$ bid truthfully and $i$ bids $\mathbf{s}'_i$ instead of his true signal vector $\mathbf{s}_i$. Let $\{T'_j\}_{j \in B}$ be the resulting allocation. Therefore, agent $i$'s utility when reporting $s'_i$ (after disregarding the $w_{-i}$ term as mentioned) is

$$\begin{aligned} v_{iT'_i}(\mathbf{s}) - p_i(\mathbf{s}'_i, \mathbf{s}_{-i}) &= g_{-iT'_i}(\mathbf{s}_{-iT'_i}) + h_{iT'_i}(\mathbf{s}_{iT'_i}) - p_i(\mathbf{s}'_i, \mathbf{s}_{-i}) \\ &= g_{-iT'_i}(\mathbf{s}_{-iT'_i}) + h_{iT'_i}(\mathbf{s}_{iT'_i}) - \left(g_{-iT'_i}(\mathbf{s}_{-iT'_i}) - g_{-iT'_i}(\mathbf{s}_{AT'_i}, \mathbf{0}_{B_{-i}T'_i}) - \sum_{j \in B \setminus \{i\}} w_{jT'_j}\right) \\ &= h_{iT'_i}(\mathbf{s}_{iT'_i}) + g_{-iT'_i}(\mathbf{s}_{AT'_i}, \mathbf{0}_{B_{-i}T'_i}) + \sum_{j \in B \setminus \{i\}} w_{jT'_j} \\ &= w_{iT'_i} + \sum_{j \in B \setminus \{i\}} w_{jT'_j} = \sum_{j \in B} w_{jT'_j} \\ &\leq \sum_{j \in B} w_{jT_j^*}, \end{aligned}$$

where $\sum_{j \in B} w_{jT_j^*}$ is $i$'s utility for bidding truthfully.

Finally, we show that the mechanism is ex post IR. Indeed, agent $i$'s utility when reporting truthfully (and without disregarding the $w_{-i}$ term) is

$$v_{iT_i^*}(\mathbf{s}_{T_i^*}) - p_i(\mathbf{s}) = \sum_{j \in B} w_{jT_j^*} - w_{-i} = \sum_{j \in B} w_{jT_j^*} - \max_{\text{partitions } \{T'_j\}} \sum_{j \in B \setminus \{i\}} w_{jT'_j} \geq 0. \quad \square$$

In the case of separable $d$-SOS valuations, the random sampling VCG is an ex post IC-IR mechanism that gives $2(d+1)$ approximation to the social welfare. The proof is identical to Theorem 3, except that Equation (5) is changed to

$$\mathbb{E}_B[w_{iT} \cdot \mathbf{1}_{i \in B}] \geq \frac{v_{iT}(\mathbf{s}_T)}{2(d+1)}$$

because we apply Lemma 2 with an arbitrary $d$.

**Remark 1.** Theorem 3 is clearly analogous to the VCG mechanism for combinatorial auctions with private values. As with VCG for private values, in many cases, there is unlikely to be a polynomial time algorithm to compute allocations and payments. Exceptions include settings we know and love such as unit-demand auctions, additive valuations, etc.

## 6. Combinatorial Auctions with Single-Dimensional Signals

In this section, we consider combinatorial valuations (general combinatorial auctions) with single-dimensional signals (as given by Definition 9).

When the signal space of each agent is of size at most $k$, we present a mechanism that gets $(k+3)$ approximation for SOS valuations (see Section 6.1) and a mechanism that gets $(2\log_2 k + 4)$ approximation for strong-SOS valuations (Definition 14; see Section 6.2 for details regarding the mechanism). For $d$-SOS and $d$-strong-SOS valuations, the mechanism generalizes to give $O(dk)$ and $O(d^2 \log k)$ approximations, respectively, as shown in Appendix C.

We first decompose the optimal welfare into two parts, OTHER and SELF. Each part will be covered by a corresponding mechanism. Let $T^* = \{T_i^*\}_{i \in [n]}$ be a welfare-maximizing allocation at signal profile $\mathbf{s}$, and let $W^*(\mathbf{s})$ be the social welfare of $T^*$ at $\mathbf{s}$. Consider the following decomposition:

$$\begin{aligned} W^*(\mathbf{s}) &= \sum_i v_{iT_i^*}(\mathbf{s}) \\ &= \sum_i v_{iT_i^*}(\mathbf{s}_{-i}, 0_i) + \sum_{i:\, s_i > 0} (v_{iT_i^*}(\mathbf{s}) - v_{iT_i^*}(\mathbf{s}_{-i}, 0_i)) \\ &\leq \sum_i v_{iT_i^*}(\mathbf{s}_{-i}, 0_i) + \sum_{i:\, s_i > 0} (v_{iT_i^*}(\mathbf{0}_{-i}, s_i) - v_{iT_i^*}(\mathbf{0})) \qquad (6) \end{aligned}$$

$$\leq \underbrace{\sum_i v_{iT_i^*}(\mathbf{s}_{-i}, 0_i)}_{\text{OTHER}} + \underbrace{\sum_{\ell=1}^{k-1} \sum_{i:\, s_i = \ell} v_{iT_i^*}(\mathbf{0}_{-i}, s_i)}_{\text{SELF}}, \qquad (7)$$

where Equation (6) follows from the definition of submodularity (and therefore, also follows the definition of strong submodularity). The last inequality follows from the nonnegativity of $v_{iT_i^*}(\mathbf{0})$. The first term in the decomposition represents the contribution of others' signals to one's value from his allocated bundle, whereas the second term represents one's contribution to his own value. Each of these terms will be targeted using a different mechanism. Whereas the OTHER term will be targeted using the same mechanism in both the SOS and strong-SOS cases, the SELF term will be treated differently.

### 6.1. ($k$+3) Approximation for SOS Valuations

Suppose $s_i \in \{0, 1, \ldots, k-1\}$ for all $i$. The mechanism is as follows.

Mechanism $k$ signals high-low ($k$-HL).

With probability $p_{RT} = (k - 1/k + 3)$, run random threshold; otherwise, run random sampling, as described.

#### 6.1.1. Mechanism Random Threshold.

- Choose a random threshold $\ell$ uniformly in $\{1, \ldots, k-1\}$.
- Let $N_{\geq \ell} = \{i : s_i \geq \ell\}$ be the "high" agents (i.e., agents with signal at least $\ell$), and let $N_{<\ell} = [n] \setminus N_{\geq \ell}$ be the "low" agents.
- For every high agent $i \in N_{\geq \ell}$ and bundle $T$, let $\overline{v}_{iT} := v_{iT}(\mathbf{s}_{N_{<\ell}}, \ell_{N_{\geq \ell}})$.
- For every low agent $i \in N_{<\ell}$ and bundle $T$, let $\overline{v}_{iT} := 0$.
- Let the allocation be

$$\overline{T} \in \arg\max_{S = \{S_i\}_{i \in N_{\geq \ell}}} \sum_{i \in N_{\geq \ell}} \overline{v}_{iS_i}.$$

(That is, the allocation maximizes the "welfare" of high agents using values $\overline{v}_{iT}$.)

- Agent $i$ that receives bundle $\overline{T}_i$ pays $v_{i\overline{T}_i}(\mathbf{s}_{-i}, s_i = \ell - 1)$.

#### 6.1.2. Mechanism Random Sampling.

- Split the agents into sets $A$ and $B$ uniformly at random.
- For each $i \in B$ and bundle $T$, let $\tilde{v}_{iT} := v_{iT}(\mathbf{s}_A, \mathbf{0}_B)$.
- For each $i \in A$ and bundle $T$, let $\tilde{v}_{iT} := 0$.
- Let the allocation be

$$\tilde{T} \in \arg\max_{S=\{S_i\}_{i\in B}} \sum_{i\in B} \tilde{v}_{iS_i}.$$

(That is, the allocation maximizes the "welfare" of agents in $B$ using values $\tilde{v}_{iT}$.)

- Charge no payments.

The $k$-HL mechanism is a random combination of two mechanisms. `Random threshold` approximates the welfare contribution of the bidders' signals to their own value (the SELF term); `random sampling` approximates the welfare contributions of the bidders' signals to other bidders' values (the OTHER term). We wish to prove the following theorem.

**Theorem 4.** *For every combinatorial auction setting with SOS valuations, single-dimensional signals, and signal space of size $k$ (i.e., $s_i \in \{0, 1, \ldots, k-1\}$ $\forall i$), mechanism $k$-*`HL` *is an ex post* IC-*IR mechanism that gives $(k+3)$ approximation to the optimal social welfare.*

We first argue that the mechanism is ex post IC-IR.

**Proof of Ex Post IC-IR.** `Random sampling` is ex post IC-IR because the agents that might receive items (agents in $B$) cannot change the allocation because their signals are ignored (and they pay nothing).

As for `random threshold`, consider a threshold $\ell$ chosen by the mechanism. If the agent's signal is below $\ell$ and the agent reports $\ell$ or above, then his payment if allocated bundle $T$ is $v_{iT}(\mathbf{s}_{-i}, s_i = \ell - 1) \geq v_{iT}(\mathbf{s})$ (i.e., the agent's utility is nonpositive). Bidding a different value below $\ell$ will grant the agent no items. If his value is $\ell$ or above, then bidding a different signal above $\ell$ will result in the same outcome because the sets $N_{\geq \ell}$ and $N_{<\ell}$ remain the same. If he bids a signal below $\ell$, then he will not receive any item, and his utility will be zero; however, bidding his true signal will result in nonnegative utility. □

In Lemma 4, we prove that `random sampling` covers the OTHER component of the social welfare, and in Lemma 3, we show that `random threshold` covers the SELF component. □

**Lemma 3.** *For SOS valuations, the* `random threshold` *mechanism gives a $(k-1)$ approximation to the* SELF *component of the optimal social welfare.*

**Proof.** Consider a threshold $\ell \in \{1, \ldots, k-1\}$ chosen in `random threshold`. Whenever $\ell$ is chosen, we have that

$$\sum_{i:\, s_i=\ell} \overline{v}_{iT_i^*} = \sum_{i:\, s_i=\ell} v_{iT_i^*}(\mathbf{s}_{N_{<\ell}}, \ell_{N_{\geq\ell}}) \geq \sum_{i:\, s_i=\ell} v_{iT_i^*}(\mathbf{0}_{-i}, s_i).$$

Because `random threshold` chooses an allocation $\overline{T} = \{\overline{T}_i\}_{i\in N_{\geq\ell}}$ that maximizes the welfare under $\overline{v}_{iT}$'s, the value of the allocation is only larger than the left expression. Because $v_{i\overline{T}_i}(\mathbf{s}) \geq \overline{v}_{i\overline{T}_i}$, we get that if $\ell$ was chosen, which happens with probability $1/k-1$, the welfare achieved is at least $\sum_{i:\, s_i=\ell} v_{iT_i^*}(\mathbf{0}_{-i}, s_i)$. Therefore, the welfare from running `random threshold` is at least

$$\sum_{\ell=1}^{k-1} \frac{1}{k-1} \sum_{i:\, s_i=\ell} v_{iT_i^*}(\mathbf{0}_{-i}, s_i) \geq \frac{\text{SELF}}{k-1}.$$

□

**Lemma 4.** *For SOS valuations, the* `random sampling` *mechanism gives a four approximation to the* OTHER *component of the optimal social welfare.*

**Proof.** Consider a set $T$. Using an application of the key Lemma 2 with respect to $v_{iT}(\mathbf{s}_{-i}, 0_i)$, we see that

$$\mathbb{E}_{A,B}[\tilde{v}_{iT}] \geq \Pr[i \in B] \cdot \mathbb{E}_{A,B}[\tilde{v}_{iT} \mid i \in B] = \frac{1}{2}\mathbb{E}_{A,B\setminus i}[\tilde{v}_{iT} \mid i \in B] \geq \frac{1}{4} v_{iT}(\mathbf{s}_{-i}, 0_i). \tag{8}$$

Therefore, the expected weight of the allocation $\{T_i^*\}_{i\in[n]}$ using weights $\tilde{v}_{iT}$'s is

$$\mathbb{E}_{A,B}\left[\sum_i \tilde{v}_{iT_i^*}\right] = \sum_i \mathbb{E}_{A,B}[\tilde{v}_{iT_i^*}] \geq \sum_i \frac{1}{4} v_{iT_i^*}(\mathbf{s}_{-i}, 0_i) = \frac{\text{OTHER}}{4}.$$

Because the mechanism chooses the optimal allocation according to the $\tilde{v}_{iT}$'s, its weight can only be larger. Moreover, because $\tilde{v}_{iT} = v_{iT}(\mathbf{s}_{-i}, 0) \le v_{iT}(\mathbf{s})$, the welfare achieved by the mechanism is at least $\mathsf{OTHER}/4$, as desired. □

We conclude by proving the claimed approximation ratio.

**Proof of Approximation.** According to Lemma 3, `random threshold` approximates $\mathsf{SELF}$ to a factor of $k-1$. According to Lemma 4, `random sampling` approximates $\mathsf{OTHER}$ to a factor of four. Therefore, running `random threshold` with probability $p_{RT}$ and `random sampling` with probability $1 - p_{RT}$ yields a welfare of

$$\begin{aligned} p_{RT}\frac{\mathsf{SELF}}{k-1} + (1-p_{RT})\frac{\mathsf{OTHER}}{4} &= \frac{k-1}{k+3}\cdot\frac{\mathsf{SELF}}{k-1} + \frac{4}{k+3}\cdot\frac{\mathsf{OTHER}}{4} \\ &= \frac{\mathsf{SELF}+\mathsf{OTHER}}{k+3} \ge \frac{W^*(\mathbf{s})}{k+3}, \end{aligned}$$

where the inequality follows Equation (7). □

### 6.2. *O*(log *k*) Approximation with Strong-SOS Valuations

Strong-SOS valuations mean the effect on the valuation is concave in one's own signal. This allows us to use a bucketing technique in order to give an $O(\log k)$ approximation to the SELF component in the decomposition depicted by Equation (7).

Consider the $\mathsf{SELF}$ term in Equation (7). We can bound this term as follows:

$$\begin{aligned} \mathsf{SELF} &= \sum_{\ell=1}^{k-1}\sum_{i:\, s_i=\ell} v_{iT_i^*}(\mathbf{0}_{-i}, s_i) \\ &= \sum_{\ell=1}^{\log_2 k}\sum_{i:\, 2^{\ell-1}\le s_i<2^{\ell}} v_{iT_i^*}(\mathbf{0}_{-i}, s_i) \\ &\le 2\sum_{\ell=1}^{\log_2 k}\sum_{i:\, 2^{\ell-1}\le s_i<2^{\ell}} v_{iT_i^*}(\mathbf{0}_{-i}, 2^{\ell-1}{}_i), \end{aligned} \tag{9}$$

where the inequality follows the definition of strong-SOS valuations.

We introduce mechanism `random bucket` to give an $O(\log k)$ approximation to the upper bound in Equation (9).

#### 6.2.1. Mechanism `Random Bucket`.

- Choose $\ell$ uniformly in $\{1, \ldots, \log_2 k\}$.
- Let $N_{B_\ell} = \{i:\ \text{such that } s_i \ge 2^{\ell-1}\}$ be the agents with signal at least $2^{\ell-1}$ and $N_{\neg B_\ell} = [n] \setminus N_{B_\ell}$.
- For $i \in N_{B_\ell}$ and bundle $T$, let $\overline{v}_{iT} := v_{iT}(\mathbf{s}_{N_{\neg B_\ell}}, \mathbf{2^{\ell-1}}_{N_{B_\ell}})$ (and $\overline{v}_{iT} := 0$ for $i \in N_{\neg B_\ell}$).
- Let the allocation be

$$\overline{T} \in \arg\max_{S=\{S_i\}_{i\in N_{B_\ell}}} \sum_{i\in N_{B_\ell}} \overline{v}_{iS_i}.$$

(That is, the allocation maximizes the "welfare" of high agents using values $\overline{v}_{iT}$.)

- Agent $i$ that receives bundle $\overline{T}_i$ pays $v_{i\overline{T}_i}(\mathbf{s}_{-i}, s_i = 2^{\ell-1} - 1)$.

We show the following approximation guarantee regarding `random bucket`.

**Lemma 5.** *For strong-SOS valuations, the* `random bucket` *mechanism is ex post IC-IR and gives a* $2\log_2 k$ *approximation to the SELF component of the optimal social welfare.*

**Proof.** The proof of ex post IC-IR is identical to that of mechanism `random threshold`, as both are threshold-based mechanisms. The proof of the approximation guarantee is also very similar to that of `random threshold`.

Consider a threshold $2^{\ell-1}$ for $\ell \in \{1, \ldots, k-1\}$ chosen in `random bucket`. Whenever $\ell$ is chosen, we have that

$$\sum_{i:\, 2^{\ell-1}\le s_i<2^{\ell}} \overline{v}_{iT_i^*} = \sum_{i:\, 2^{\ell-1}\le s_i<2^{\ell}} v_{iT_i^*}(\mathbf{s}_{N_{\neg B_\ell}}, \mathbf{2^{\ell-1}}_{N_{B_\ell}}) \ge \sum_{i:\, 2^{\ell-1}\le s_i<2^{\ell}} v_{iT_i^*}(\mathbf{0}_{-i}, 2^{\ell-1}{}_i).$$

Because `random bucket` chooses an allocation that maximizes the $\overline{v}_{iT}$'s, the value of the allocation is only larger. Because $v_{i\overline{T}_i}(\mathbf{s}) \geq \overline{v}_{i\overline{T}_i}$, we get that if $\ell$ was chosen, which happens with probability $1/\log_2 k$, the welfare achieved is at least $\sum_{i:\, 2^{\ell-1} \leq s_i < 2^\ell} v_{iT_i^*}(\mathbf{0}_{-i}, 2^{\ell-1}{}_i)$. Therefore, the welfare from running `random bucket` is at least

$$\sum_{\ell=1}^{\log_2 k} \frac{1}{\log_2 k} \sum_{i:\, 2^{\ell-1} \leq s_i < 2^\ell} v_{iT_i^*}(\mathbf{0}_{-i}, 2^{\ell-1}{}_i) \geq \frac{\mathsf{SELF}}{2\log_2 k}.$$

□

Mechanism $k$-`signals strong submodular over signals` ($k$-`SS`) runs `random bucket` with probability $p_{RB} = \frac{\log_2 k}{\log_2 k + 2}$ and mechanism `random sampling` with probability $1 - p_{RB}$.

**Theorem 5.** *For every combinatorial auction with single-dimensional signals with strong-SOS valuations and signal space of size $k$ (i.e., $s_i \in \{0, 1, \ldots, k-1\}\ \forall i$), mechanism $k$-`SS` is ex post IC-IR and gives $(2\log_2 k + 4)$ approximation to the optimal social welfare.*

**Proof.** We already established that both `random bucket` and `random sampling` are ex post IC-IR; hence, $k$-`SS` is ex post IC-IR as well. As for the approximation, according to Lemma 5, with probability $p_{RB}$ we get $2\log_2 k$ approximation to SELF, and according to Lemma 4, with probability $1 - p_{RB}$, we get a four approximation to OTHER. Overall, the expected welfare is at least

$$p_{RB}\frac{\mathsf{SELF}}{2\log_2 k} + (1 - p_{RB})\frac{\mathsf{OTHER}}{4} = \frac{\mathsf{SELF} + \mathsf{OTHER}}{2\log_2 k + 4} \geq \frac{W^*}{2\log_2 k + 4},$$

as desired. □

## 7. Open Problems

Our analysis and results suggest many open problems.

- For combinatorial auctions with multidimensional signals, is separability a necessary condition for achieving constant approximation to welfare? This problem is open even for single-dimensional signals and even for "simple" combinatorial valuations, such as unit demand.
- For single-parameter SOS valuations, downward-closed feasibility, and single-dimensional signals, closing the gap between $1/4$ and $1/2$ is open. We note that recently Amer and Talgam-Cohen [2] showed that $1/2$ is the correct answer for binary signals and matroid feasibility constraints.
- The exact same gap applies for combinatorial separable-SOS valuations with multidimensional signals.
- How does the distinction between SOS and strong SOS affect the problems above, if at all?
- When considering the relaxation of SOS valuations to $d$-SOS valuations, there is a gap between the positive and negative results with respect to the dependence on $d$.

More generally, what other classes of valuations give rise to approximately efficient mechanisms in settings with interdependent valuations?

### Appendix A. Unit-Demand Valuations with Single Crossing

Even though single crossing is a strong-enough condition to implement the fully efficient mechanism in a variety of single-parameter environments, generalizations of this condition fail even in the simplest multiparameter environments. We consider the case where bidders are unit demand and each bidder has a scalar as a signal. We define single crossing for this setting as follows.

**Definition A.1** (Single Crossing for Unit-Demand Valuations). A valuation profile $\mathbf{v}$ is said to be single crossing if for every agent $i$, signals $\mathbf{s}_{-i}$, item $j$, and agent $\ell$,

$$\frac{\partial}{\partial s_i} v_{ij}(\mathbf{s}_{-i}, s_i) \geq \frac{\partial}{\partial s_i} v_{\ell j}(\mathbf{s}_{-i}, s_i). \tag{A.1}$$

In this section, we show that in the case two nonidentical items are for sale and the valuations are unit demand and satisfy single crossing as defined in Equation (A.1), any truthful mechanism is bounded away from achieving full efficiency.

In order to give the lower bound, we first give a characterization of ex post IC and IR mechanisms in multidimensional environments in interdependent values settings (Appendix A.1). We then turn to prove the lower bound (Appendix A.2).

## A.1. Cycle Monotonicity

In the IPV model, Rochet [35] introduced cycle monotonicity as a necessary and sufficient condition on the allocation to be implementable in dominant strategies (DSIC) for multidimensional environments. It was noticed that a straightforward analogue holds for the IDV value model for ex post implementability (EPIC) (in Vohra [41], this fact is stated without a proof).

Fix a feasible allocation rule $\mathbf{x} = \{x_i\}_{i\in[n]}$, where $x_{iT}(\mathbf{s})$ is the probability agent $i$ receives a bundle $T$ under bid profile $\mathbf{s}$. For each agent $i$, consider the graph $G_i^{\mathbf{x}}$, where there is a vertex for each signal profile $\mathbf{s}$ and there is a directed edge from $\mathbf{s}$ to $\mathbf{t}$ if $\mathbf{s}_{-i} = \mathbf{t}_{-i}$. The weight of edge $(\mathbf{s},\mathbf{t})$ is

$$w(\mathbf{s},\mathbf{t}) = \mathbb{E}_{T\sim x_i(\mathbf{s})}[v_{iT}(\mathbf{s})] - \mathbb{E}_{T\sim x_i(\mathbf{t})}[v_{iT}(\mathbf{s})] = \sum_{T\subseteq[m]} x_{iT}(\mathbf{s})v_{iT}(\mathbf{s}) - \sum_{T\subseteq[m]} x_{iT}(\mathbf{t})v_{iT}(\mathbf{s}).$$

The following theorem states that a necessary and sufficient condition for ex post implementability of $\mathbf{x}$ is that for every agent $i$, every directed cycle in $G_i^{\mathbf{x}}$ is nonnegative. The proof is a straightforward adjustment of the original proof in Rochet [35] and is given for completeness.

**Theorem A.1.** *The allocation rule* $\mathbf{x}$ *is implementable by an ex post IC mechanism if and only if for every agent* $i$*, all directed cycles in* $G_i^{\mathbf{x}}$ *have nonnegative weight.*

**Proof.** We first show that if the allocation rule is implementable, then there are no negative cycles. Fix some payment rule $\mathbf{p} = \{p_i\}_{i\in[n]}$, where $p_i(\mathbf{s})$ is the payment of agent $i$ under bid profile $\mathbf{s}$. Let $\mathbf{s}_{-i}$ be the real signals of all bidders except $i$, and consider a cycle $\mathbf{s}^1 \to \mathbf{s}^2 \to \cdots \to \mathbf{s}^\ell \to \mathbf{s}^1$ in $G_i^{\mathbf{x}}$, where $\mathbf{s}^t = (\mathbf{s}_{-i}, s_i = \zeta_t)$ for $t \in [\ell]$. Because $(\mathbf{x},\mathbf{p})$ is an ex post IC mechanism, for every true signal $s_i = s$, agent $i$ is at least as well off bidding $s$ than any other bid $s'$. We get that

$$\begin{gathered}\mathbb{E}_{T\sim x_i(\mathbf{s}^1)}[v_{iT}(\mathbf{s}^1)] - p_i(\mathbf{s}^1) \ge \mathbb{E}_{T\sim x_i(\mathbf{s}^2)}[v_{iT}(\mathbf{s}^1)] - p_i(\mathbf{s}^2)\\ \vdots \\ \mathbb{E}_{T\sim x_i(\mathbf{s}^{\ell-1})}[v_{iT}(\mathbf{s}^{\ell-1})] - p_i(\mathbf{s}^{\ell-1}) \ge \mathbb{E}_{T\sim x_i(\mathbf{s}^{\ell})}[v_{iT}(\mathbf{s}^{\ell-1})] - p_i(\mathbf{s}^{\ell})\\ \mathbb{E}_{T\sim x_i(\mathbf{s}^{\ell})}[v_{iT}(\mathbf{s}^{\ell})] - p_i(\mathbf{s}^{\ell}) \ge \mathbb{E}_{T\sim x_i(\mathbf{s}^1)}[v_{iT}(\mathbf{s}^{\ell})] - p_i(\mathbf{s}^1).\end{gathered}$$

Summing over the inequalities and using the convention that $\ell + 1 = 1$, we get that

$$\begin{aligned}&\sum_{j=1}^{\ell}\mathbb{E}_{T\sim x_i(\mathbf{s}^j)}[v_{iT}(\mathbf{s}^j)] - \sum_{j=1}^{\ell}p_i(\mathbf{s}^j) \ge \sum_{j=1}^{\ell}\mathbb{E}_{T\sim x_i(\mathbf{s}^{j+1})}[v_{iT}(\mathbf{s}^j)] - \sum_{j=1}^{\ell}p_i(\mathbf{s}^j)\\ &\iff \sum_{j=1}^{\ell}(\mathbb{E}_{T\sim x_i(\mathbf{s}^j)}[v_{iT}(\mathbf{s}^j)] - \mathbb{E}_{T\sim x_i(\mathbf{s}^{j+1})}[v_{iT}(\mathbf{s}^j)]) \ge 0,\end{aligned}$$

where the left hand side of the last inequality is exactly the weight of the cycle.

We now show how to compute payments that implement a given allocation rule $\mathbf{x}$ that induces no negative cycles for any $i$ and $G_i^{\mathbf{x}}$. Given $G_i^{\mathbf{x}}$, one can compute payments as follows.

- Add a dummy node $d$ with edges of weight zero to all nodes in $G_i^{\mathbf{x}}$.
- For every node $\mathbf{s}$ of $G_i^{\mathbf{x}}$, let $\delta(\mathbf{s})$ be the distance of the shortest path from $d$ to $\mathbf{s}$.
- Set $p_i(\mathbf{s}) = -\delta(\mathbf{s})$.

Fix signals of the other players $\mathbf{s}_{-i}$. Let $s$ be player $i$'s true signal and $s'$ be some other possible signal for $i$. Denote $\mathbf{s} = (\mathbf{s}_{-i}, s)$ and $\mathbf{s}' = (\mathbf{s}_{-i}, s')$. Consider the nodes $\mathbf{s}$ and $\mathbf{s}'$ in $G_i^{\mathbf{x}}$. Because $\delta(\mathbf{s}')$ is the length of the shortest path from $d$, it must be that

$$\delta(\mathbf{s}') \le \delta(\mathbf{s}) + w(\mathbf{s},\mathbf{s}'),$$

where $w(\mathbf{s},\mathbf{s}')$ is the weight of the edge from $\mathbf{s}$ to $\mathbf{s}'$. Substituting $w(\mathbf{s},\mathbf{s}') = \mathbb{E}_{T\sim x_i(\mathbf{s})}[v_{iT}(\mathbf{s})] - \mathbb{E}_{T\sim x_i(\mathbf{s}')}[v_{iT}(\mathbf{s})]$, $p_i(\mathbf{s}) = -\delta(\mathbf{s})$, and $p_i(\mathbf{s}') = -\delta(\mathbf{s}')$, we get

$$\mathbb{E}_{T\sim x_i(\mathbf{s})}[v_{iT}(\mathbf{s})] - p_i(\mathbf{s}) \ge \mathbb{E}_{T\sim x_i(\mathbf{s}')}[v_{iT}(\mathbf{s})] - p_i(\mathbf{s}'),$$

as desired. □

## A.2. Lower Bounds for Deterministic and Randomized Mechanisms

**Proposition A.1.** *There exists a setting with two items and two agents with unit-demand and single-crossing valuations, such that no deterministic truthful mechanism achieves more than* $1/2$ *of the optimal welfare.*

**Proof.** Consider the setting depicted in Figure A.1, with two agents, 1 and 2, and two items, $a$ and $b$. $s_1 \in \{0,1\}$, and $s_2$ is fixed. The values at $s_1 = 0$ are

$$v_{1a}(0) = 1, v_{1b}(0) = 0, v_{2a}(0) = 0, v_{1b}(0) = 1,$$

and at $s_1 = 1$, they are

$$v_{1a}(1) = 1 + H + \epsilon, v_{1b}(1) = H, v_{2a}(1) = H, v_{1b}(1) = 1,$$

for some arbitrarily large $H$ and a sufficiently small $\epsilon$. One can easily verify that the valuations satisfy Equation (A.1) and hence, single crossing; indeed, when agent 1's signal increases, the valuation of agent 1 for each one of the item increases by more than the change in agent 2's valuation.

We show that no deterministic truthful mechanism can get better than two approximation. In order to get better than two approximation, the mechanism must allocate item $a$ to agent 1 and item $b$ to bidder 2 at signal $s_1 = 0$. At $s_1 = 1$, allocating item $b$ to agent 1 and item $a$ to agent 2 obtains a welfare of $2H$, whereas any other allocation obtains at most a welfare of $H + 2 + \epsilon$. Because $H$ can be arbitrarily large, one must allocate item $b$ to agent 1 and item $a$ to agent 2 at signal $s_1 = 1$ in order to get an approximation ratio better than two. Consider such an allocation rule $\mathbf{x}$ and the graph $G_1^{\mathbf{x}}$. This graph has one cycle, with one edge from $s_1 = 0$ to $s_1 = 1$ and one edge from $s_1 = 1$ to $s_1 = 0$. The weight of this cycle is

$$(v_{1a}(0) - v_{1b}(0)) + (v_{1b}(1) - v_{1a}(1)) = (1 - 0) + (H - (H + 1 + \epsilon)) = -\epsilon < 0.$$

Based on Theorem A.1, this implies that this allocation rule is not implementable. □

**Proposition A.2.** *There exists a setting with two items and two agents with unit-demand and single-crossing valuations, such that no randomized truthful mechanism achieves more than* $\sqrt{2} + 2/4$ *of the optimal welfare.*

**Proof.** Consider the setting depicted in Figure A.2, with two agents, 1 and 2, and two items, $a$ and $b$. $s_1 \in \{0, 1\}$, and $s_2 \in \{0, 1\}$. The values are

$$\begin{array}{llll} v_{1a}(0,0) = 1, & v_{1b}(0,0) = 0, & v_{2a}(0,0) = 0, & v_{1b}(0,0) = 1, \\ v_{1a}(1,0) = 1 + \sqrt{2}H, & v_{1b}(1,0) = H, & v_{2a}(1,0) = H, & v_{1b}(1,0) = 1, \\ v_{1a}(0,1) = 1, & v_{1b}(0,1) = H, & v_{2a}(0,1) = H, & v_{2b}(0,1) = 1 + \sqrt{2}H, \\ v_{1a}(1,1) = 1 + \sqrt{2}H, & v_{1b}(1,1) = H, & v_{2a}(1,1) = H, & v_{2b}(1,1) = 1 + \sqrt{2}H, \end{array}$$

for an arbitrarily large $H$. One can easily verify that the valuations are single crossing. We claim that the following equalities hold with respect to the allocation rule of the optimal randomized mechanism.

a. For every $s_1$, $s_2$, $x_{1a}(s_1, s_2) = x_{2b}(s_2, s_1)$ and $x_{2a}(s_1, s_2) = x_{1b}(s_2, s_1)$.
b. For some $q \in [0, 1]$, $x_{1a}(0, 0) = x_{2b}(0, 0) = q$ and $x_{1\emptyset}(0, 0) = x_{2\emptyset}(0, 0) = 1 - q$.
c. For some $p \in [0, 1]$, $x_{1a}(0, 1) = p$ and $x_{1b}(0, 1) = 1 - p$.

We next prove the equalities.

a. Consider some implementable allocation rule $\bar{x}$, and consider the allocation rule $\tilde{x}$ where $\tilde{x}_{1a}(s_1, s_2) = \bar{x}_{2b}(s_2, s_1)$ and $\tilde{x}_{2a}(s_1, s_2) = \bar{x}_{1b}(s_2, s_1)$ for every $s_1$, $s_2$. Note that the valuations are symmetric (i.e., the role of item $a$ ($b$) for agent 1 is the same as the role of items $b$ ($a$) for agent 2). By symmetry, $\bar{x}$ is implementable if and only if $\tilde{x}$ is implementable, and both allocation rules have the same approximation guarantee. Clearly, an allocation rule $x$ that applies allocation rules $\bar{x}$ and $\tilde{x}$, with probability $\frac{1}{2}$ each, maintains the same approximation guarantee. Moreover, this allocation rule satisfies the desired property.

b. The optimal mechanism gains nothing from assigning any positive probability for allocating item $b$ to agent 1 under signal profile (0, 0). This is because item $b$ grants no value to agent 1, and in terms of incentives, it can only incentivize agent 1 to misreport his signal at signal profile (1, 0). Analogously, the optimal mechanism gains nothing from assigning any positive probability for allocating item $a$ to agent 2 under signal profile (0, 0). By (a), $x_{1a}(0, 0) = x_{2b}(0, 0) = q$ for some $q \in [0, 1]$. To conclude the proof of (b), note that the only other feasible set for the agents is the empty set (otherwise, agent 1 has some probability to get item $b$, and agent 2 has some probability to get item $a$).

c. Consider $G_1^{\mathbf{x}}$ and the cycle $C = (0, 0) \to (1, 0) \to (0, 0)$ in $G_1^{\mathbf{x}}$. This is the only cycle that contains the node (1, 0) in $G_1^{\mathbf{x}}$. Assume $x_{1\emptyset}(1, 0) > 0$.

d. Transferring $z \in (0, 1]$ probability from $x_{1\emptyset}(1, 0)$ to $x_{1a}(1, 0)$ decreases the weight of the edge $(0, 0) \to (1, 0)$ by $z$ and increases the weight of the edge $(1, 0) \to (0, 0)$ by $z(1 + \sqrt{2}H) > z$. Therefore, its net effect on the weight of $C$ is positive. Transferring $z \in (0, 1]$ probability from $x_{1\emptyset}(1, 0)$ to $x_{1b}(1, 0)$ does not affect the weight of the edge $(0, 0) \to (1, 0)$ and increases the weight of the edge $(1, 0) \to (0, 0)$ by $zH$. Therefore, its net effect on the weight of $C$ is positive. Because transferring $x_{1\emptyset}(1, 0)$ to $x_{1a}(1, 0)$ and

**Figure A.1.** An instance with unit-demand single-crossing valuations where no deterministic truthful allocation achieves more than a half of the optimal welfare.

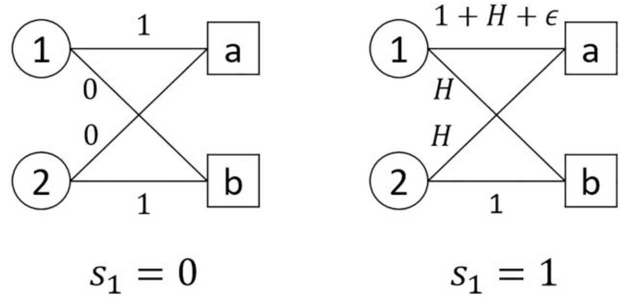
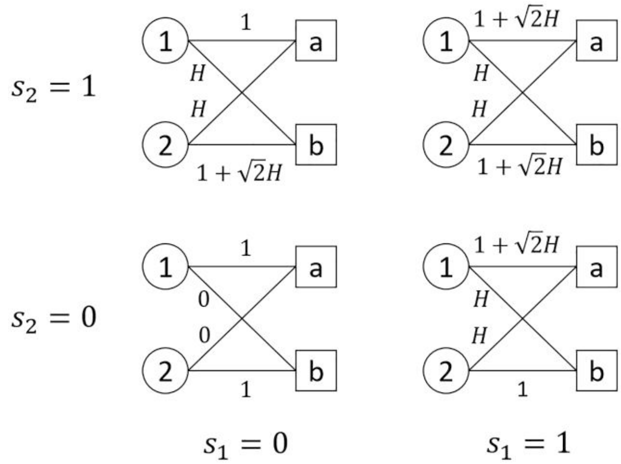


**Figure A.2.** An instance with unit-demand single-crossing valuations where no randomized truthful allocation achieves more than $\sqrt{2}+2/4$ of the optimal welfare.




$x_{1b}(1,0)$ increases welfare and does not violate cycle monotonicity, the optimal mechanism clearly assigns no probability to $x_{1\emptyset}(1,0)$.

Now, assume $x_{1\{a,b\}}(1,0) > 0$. By moving this probability to $x_{1a}(1,0)$, we get the same expected welfare at (1,0), and the weight of the edges in $C$ does not change. Therefore, we may also assume the mechanism does not assign positive utility to $x_{1\{a,b\}}(1,0)$.

According to Theorem A.1, in any truthful mechanism, the weight of the cycle $C$ must be nonnegative. This translates to the following condition:

$$\begin{aligned}&(\mathbb{E}_{T\sim x_1(0,0)}[v_{1T}(0,0)] - \mathbb{E}_{T\sim x_1(1,0)}[v_{1T}(0,0)]) - (\mathbb{E}_{T\sim x_1(1,0)}[v_{1T}(1,0)] - \mathbb{E}_{T\sim x_1(0,0)}[v_{1T}(1,0)])\\ &= (q-p) + (p(1+\sqrt{2}H) + (1-p)H - q(1+\sqrt{2}H)) \geq 0\\ &\Rightarrow q \leq p(1-1/\sqrt{2}) + \frac{1}{\sqrt{2}}.\end{aligned}$$

In the optimal mechanism, $q$ will be as large as possible in order to maximize the expected welfare at signal profile (0, 0). Hence, we can assume $q = p(1-1/\sqrt{2}) + 1/\sqrt{2}$. Therefore, the approximation ratio at profile (0, 0) is at most $q = p(1-1/\sqrt{2}) + 1/\sqrt{2}$. At profile (0, 1), if item $a$ is allocated to agent 1 (which happens with probability $p$), the welfare of the mechanism is at most $2+\sqrt{2}H$, whereas the welfare of the optimal allocation is $2H$. As $H$ can be arbitrarily large, this approximation ratio tends to $\frac{1}{\sqrt{2}}$. Therefore, the approximation ratio at profile (1,0) is at most $\frac{p}{\sqrt{2}} + (1-p) = 1 - p(1-1/\sqrt{2})$. he optimal mechanism would balance between the approximation ratio at (0, 0) and at (1, 0); therefore, it uses $p$ that solves

$$p\left(1-\frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}} = 1 - p\left(1-\frac{1}{\sqrt{2}}\right).$$

Solving for $p$, we get $p = 1/2$. This leads to an approximation ratio of at most $\frac{2+\sqrt{2}}{4}$, as promised. □

## Appendix B. $n-1$ Lower Bound for Deterministic Mechanisms with Single-Crossing SOS Valuations

We show that for downward-closed environments, even if valuations satisfy a single-crossing condition and are SOS, any deterministic mechanism cannot obtain a better approximation to the optimal welfare than $n-1$.

**Proposition B.1.** *There exists a downward-closed environment with valuations that satisfy single crossing for which no deterministic mechanism can get a better approximation than* $n-1$ *to the optimal social welfare.*

**Proof.** Consider a set of $n$ bidders, where $\mathcal{I} = \{1\} \cup P(\{2,\ldots,n\})$ and $P(\{2,\ldots,n\})$ is the power set of the set $\{2,\ldots,n\}$. Only agent 1 has a signal $s_1 \in \{0,1\}$, and other players do not have signals. The valuations are

$$\begin{aligned}&v_1(0) = 1 \qquad v_1(1) = 1+H\\ &v_i(0) = 0 \qquad v_i(1) = H \qquad \forall i \in \{2,\ldots,n\}\end{aligned}$$

for an arbitrary large value $H \gg 1$. One can easily verify that these valuations satisfy single crossing and SOS.

Any deterministic mechanism that wants to get any approximation to the social welfare must allocate to agent 1 when $s_1 = 0$. In addition, if a deterministic mechanism wants to get a better approximation than $n-1$ to the optimal social welfare, agent 1 cannot be allocated when $s_1 = 1$. Otherwise, none of the bidders in $\{2,\ldots n\}$ can get allocated because the only set in

$\mathcal{I}$ that contains agent 1 is the singleton set. Therefore, if agent 1 is allocated at $s_1 = 1$, the achieved welfare is $1 + H$, whereas the optimal welfare is $(n-1) \cdot H$ (when serving all agents in $\{2, \ldots, n\}$). For an arbitrary large $H$, this ratio approaches $n-1$.

The proof follows because serving agent 1 at $s_1 = 0$ and not serving agent 1 at $s_1 = 1$ violate monotonicity. □

**Remark B.1.** The $n-1$ factor is tight for single-crossing valuations. If $[n] \in \mathcal{I}$, then the mechanism can always allocate all agents. Otherwise, one can always allocate only to the highest-valued agent, which is monotone because of single crossing. Because the largest feasible set is of size at most $n-1$ in this case, allocating to the highest-valued agent yields an approximation ratio of $n-1$.

**Proposition B.2.** *There exists a combinatorial auctions environment with valuations that satisfy single crossing for which no deterministic mechanism can get a better approximation than $n-1$ to the optimal social welfare.*

**Proof.** Consider a set of $n$ bidders and $n-1$ items. Only agent $n$ has a binary signal. The valuation function of $n$ for item $j$ is $v_{nj}(s_n) = H \cdot s_n$ and for the set of all $n-1$ items is $v_{1[n-1]}(s_n) = 1 + (H + \epsilon) \cdot s_n$. Agent $n$ is unit demand for every strict subset of items (that is, their value for a nonempty subset of items $T \subset [n-1]$ is $v_{nT}(s_n) = H \cdot s_n$).

Every agent $i \neq n$ is a single-minded agent, where their value for item $i$ is $v_{ii}(s_n) = H \cdot s_n$, and for every subset $T \not\ni i$, $v_{iT}(s_n) = 0$. It is easy to verify that the valuations satisfy single crossing and submodularity over signals.

Consider first the case where $s_n = 0$. Every deterministic mechanism *must* allocate all items to agent $n$ because this is the only deterministic allocation that has any value. By individual rationality, agent $n$ pays at most one. Now consider the case where $s_n = 1$. In this case, we claim that all items should be allocated to agent $n$ when they report $b_n = 1$.

Suppose toward a contradiction that only a strict subset of items $T \subset [n-1]$ is allocated to agent $n$ when they report $b_n = 1$. In this case, $n$'s value from the allocation is $v_{nT}(1) \leq H$, and their utility is at most $H$ as well. Now suppose that they report $b_n = 0$ instead. In this case, as we claimed, they must be allocated all items and pay at most one. Therefore, their utility is at least $v_{n[n-1]}(1) - 1 = H + \epsilon$. Hence, they get a higher utility by misreporting their signal, contradicting truthfulness.

We get that when $s_n = 1$, all items must be allocated to agent 1, achieving welfare $H + 1 + \epsilon$, whereas an optimal allocation allocates item $i$ to agent $i$, obtaining welfare $(n-1)H$. As $H$ grows large, this implies that no EPIC IR mechanism can get a better approximation than $n-1$ to the optimal social welfare. □

## Appendix C. Results for *d*-SOS

We now extend the results in Section 6 to the case of combinatorial $d$-SOS and combinatorial $d$-strong-SOS valuations with single-dimensional signals. We first note that if we consider $d$-SOS valuations, then Equation (6) in the decomposition becomes

$$\begin{aligned} W^* &\leq \sum_i v_{iT_i^*}(\mathbf{s}_{-i}, 0_i) + \sum_{i:\, s_i > 0} d \cdot (v_{iT_i^*}(\mathbf{0}_{-i}, s_i) - v_{iT_i^*}(\mathbf{0})) \\ &\leq \underbrace{\sum_i v_{iT_i^*}(\mathbf{s}_{-i}, 0_i)}_{\text{OTHER}} + \underbrace{\sum_{\ell=1}^{k-1} \sum_{i:\, s_i = \ell} d \cdot v_{iT_i^*}(\mathbf{0}_{-i}, s_i)}_{\text{SELF}}, \end{aligned} \tag{C.1}$$

and we now show the extension of Theorem 4 to $d$-SOS valuations.

**Theorem C.1.** *For every combinatorial auction with d-SOS valuations over single-dimensional signals and signal space of size k* (*i.e.*, $s_i \in \{0, 1, \ldots, k-1\}$ $\forall i$), *there exists a truthful mechanism that gives $d(k+1) + 2$ approximation to the optimal social welfare.*

**Proof.** The mechanism is identical to $k$-`HL` but runs (`random threshold`) with probability $p_{RT} = ((k-1)d/d(k+1)) + 2$ and (`random sampling`) with probability $1 - p_{RT}$. The mechanism was already proved to be truthful in Section 6.1.

`Random threshold` now gives a $d(k-1)$ approximation to the new SELF term. The proof is the same as of Lemma 3, but the extra factor of $d$ comes from the fact the new SELF term is $d$ times larger.

`Random sampling` gives a $2(d+1)$ approximation to the OTHER term. Although this term is the same for $d$-SOS, the new factor is because of the fact that when applying Lemma 2 in the proof of Lemma 4, we get that $\mathbb{E}_{A,B}[\tilde{v}_{iT}] \geq 1/2(d+1)$ $v_{iT}(\mathbf{s}_{-i}, 0_i)$ instead of the bound that we get in Equation (8).

The approximation for $d$-SOS valuations follows from the new decomposition, the approximation guarantees the mechanisms get for the terms of the decomposition, and the updated probability $p_{RT}$. □

We next extend Theorem 5.

**Theorem C.2.** *For every combinatorial auction with d-strong-SOS valuations over single-dimensional signals and signal space of size k* (*i.e.*, $s_i \in \{0, 1, \ldots, k-1\}$ $\forall i$), *there exists a truthful mechanism that gives* $(d(d+1)\log_2 k + 2(d+1))$ *approximation to the optimal social welfare.*

**Proof.** The mechanism is identical to mechanism $k$-`SS` from Section 6.2 but runs `random bucket` with probability $p_{RB} = (d \log_2 k)/(d \log_2 k + 2)$ and (`random sampling`) with probability $1 - p_{RB}$.

The SELF term from Equation (9) is now bounded via the following:

$$\begin{aligned}\mathsf{SELF} &= \sum_{\ell=1}^{k-1}\sum_{i:\, s_i=\ell} d\cdot v_{iT_i^*}(\mathbf{0}_{-i}, s_i)\\ &= \sum_{\ell=1}^{\log_2 k}\sum_{i:\, 2^{\ell-1}\le s_i<2^\ell} d\cdot v_{iT_i^*}(\mathbf{0}_{-i}, s_i)\\ &\le \sum_{\ell=1}^{\log_2 k}\sum_{i:\, 2^{\ell-1}\le s_i<2^\ell} d(d+1)\cdot v_{iT_i^*}(\mathbf{0}_{-i}, 2^{\ell-1}{}_i),\end{aligned} \tag{C.2}$$

where the inequality follows the definition of $d$-strong-SOS valuations.

The new bound changes the guarantee of `random bucket` to get a $d(d+1)\log_2 k$ approximation to the SELF term, where the proof is identical to that of Lemma 5.

As stated in Theorem C.1, `random sampling` approximates the OTHER term to a factor $2(d+1)$. The proof of the new bound follows the new decomposition, the updated probabilities, and the new approximation guarantees of the mechanisms being run. □

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